Homework 4, Differential Geometry
due 2/24/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. We first construct some "smoothing functions":

(i) Show that the function
$$f(x) = \begin{cases} 0 & x \leq 0 \\ \exp(-1/x) & x > 0 \end{cases}$$
is smooth on $\mathbb{R}$. Draw its graph.

(ii) Conclude that the function $g(x) = f(1 - x^2)$ is smooth on $\mathbb{R}$ and draw its graph.

(iii) Conclude that the function $h(x) = \int_0^x g(t)dt$ is smooth on $\mathbb{R}$ and draw its graph.

Problem 2. For $a < b$ and $\alpha < \beta$ real numbers show that there is a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ with the following property:
$$f(x) = \begin{cases} \alpha & x \leq a \\ \beta & x \geq b \\ \text{monoton increasing} & a < x < b \end{cases}$$
Draw the graph of f. Further show that there is an analogous smooth function which interpolates two constant functions of values $\alpha > \beta$ in a monoton decreasing manner.

Problem 3. For $a_1 < a_2 < b_2 < b_1$ and $c > 0$ real numbers show that there exists a smooth (bump) function $f\colon \mathbb{R} \to \mathbb{R}$ with the following property:
$$f(x) = \begin{cases} 0 & x \leq a_1 \text{ and } x \geq b_1 \\ c & a_2 \leq x \leq b_2 \\ \text{monoton increasing} & a_1 < x < a_2 \\ \text{monoton decreasing} & b_2 < x < b_1 \end{cases}$$
Draw the graph of f.

Problem 4. Show that regular homotopy of regular curves $\gamma\colon I \to \mathbb{R}^n$ is an equivalence relation, that is:

(i) $\gamma \sim \gamma$ (where the symbol $\sim$ stands for "regularly homotopic");

(ii) $\gamma \sim \tilde{\gamma}$ implies $\tilde{\gamma} \sim \gamma$;

(iii) $\gamma \sim \tilde{\gamma}$ and $\tilde{\gamma} \sim \hat{\gamma}$ implies $\gamma \sim \hat{\gamma}$ (here you have to use a smoothing function).

Problem 5. Show that the curve $\gamma(t) = (t, |t|)$, $t \in \mathbb{R}$ can be smoothly parametrized. Can you parametrize so that the curve becomes regular?

Problem 6. Let γ be a regular closed curve in $\mathbb{R}^n$. Show that there is a regular homotopy Γ through closed curves with $\Gamma(-, 0) = \gamma$ and $\Gamma(-, 1)$ an arclength parametrization of γ.

Problem 7. Let $f\colon \mathbb{R} \to \mathbb{R}^n$ be a smooth function. Give necessary and sufficient conditions on f so that the antiderivative $F(x) = \int_0^x f(t)dt$ is periodic with period $p \neq 0$.