Homework 3, Differential Geometry
due 2/17/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. Show that every matrix $A \in \mathbf{O}(2, \mathbb{R})$ is of the form $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ or $JR(\alpha)$. Interpret the maps $x \mapsto R(\alpha)x$ and $x \mapsto JR(\alpha)x$ for $x \in \mathbb{R}^2$.

Problem 2. Show that the set $\mathbf{GL}(n, \mathbb{R})$ of invertible matrices forms a group under matrix multiplication. Show the same for the orthogonal group $\mathbf{O}(n, \mathbb{R})$ and the special orthogonal group $\mathbf{SO}(n, \mathbb{R})$.

Problem 3. Identify $\mathbb{C} = \mathbb{R}^2$ in the usual way $z = x_1 + ix_2 = (x_1, x_2)$. If $a \in \mathbb{C}$ show that the complex multiplication map $z \to az$ corresponds to the map $x \mapsto |a|R(\alpha)x$ for $x = (x_1, x_2)$, where $a = |a|e^{i\alpha}$ is the polar form of the complex number a. Interpret the map geometrically. What is $R(\alpha)$ for $a = i$?

Problem 4. Let $f\colon I \to \mathbb{C}$ be a smooth complex valued function and $t_0 \in I$ fixed.

(i) Show that the initial value problem
$$z'(t) = f(t)z(t) \qquad z(t_0) = z_0 \in \mathbb{C}$$
has the unique solution $z(t) = z_0 \exp(\int_{t_0}^t f(s)ds)$. *Hint* : for uniqueness let $w(t)$ be another solution of the same initial value problem and contemplate the expression $w(t)/z(t)$.

(ii) Show that if $f\colon I \to i\mathbb{R}$ is imaginary valued, the length of the solution $z(t)$ is constant, i.e. $|z(t)| = |z_0|$. Does the converse also hold, i.e. if you know that the length of the solution to the ODE $z' = fz$ is preserved, then f necessarily has to be imaginary valued.

(iii) Show that if $z_1(t)$, $z_2(t)$ solve the ODE $z'(t) = f(t)z(t)$, then $z_1(t) = cz_2(t)$ for some $c \in \mathbb{C}$ (this is another way to state the uniqueness property).

Problem 5. Provide a complete proof that a regular plane curve $\gamma\colon I \to \mathbb{R}^2$ can near each point $\gamma(t_0)$ be written as a graph over the tangent line: more precisely, there exists a smooth real valued map $x \to f(x)$ for small x with $f(0) = 0$ so that $x \mapsto xT(t_0) + f(x)JT(t_0)$ parametrizes γ near $\gamma(t_0)$. Here $T = \gamma'/||\gamma'||$ is the unit length tangent vector.

Problem 6. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a regular (smooth) closed curve with period p. Show that there exist an orientation preserving diffeomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$, a number $\tilde{p} \in \mathbb{R}$ such that $\varphi(s + \tilde{p}) = \varphi(s) + p$ and $\tilde{\gamma} = \gamma \circ \varphi$ is an arclength parametrized closed curve with period $\tilde{p}$.

Problem 7. A regular curve $\gamma\colon I \to \mathbb{R}^2$ is called *convex* if for all $t \in I$ the curve always lays to one side of its tangent line at $\gamma(t)$. Show that for a convex curve its curvature $\kappa(t)$ never changes sign.

Problem 8. Let $\gamma\colon [0, L] \to \mathbb{R}^n$ be arclength parametrized. Show that the distance between the endpoints of the curve can at most be L, and equality can only hold when γ is a straight line segment. Thus, the shortest path between two points is the straight line segment connecting them.