Final Exam, Differential Geometry
due 5/5/17

Problem 1. If you have read Riemann's thesis, you probably came across the following statement: if the curvature of a Riemannian manifold vanishes identically (such Riemannian manifolds are called flat) then there exists an isometry of the Riemannian manifold with Euclidean space (at least locally near each point), i.e. a flat Riemannian metric is locally isometric to the standard inner product on $\mathbb{R}^n$. This first problem asks you to give a proof of this statement in the case $n = 2$. Here the precise statement:

Let (U, g) be a 2-dimensional (local) Riemannian manifold whose curvature function $K_g \equiv 0$ is the zero function. Then, to each point $p_0 \in U$ there exists an open disk $B \subset U$ centered at p_0 and a diffeomorphism $\varphi\colon B \to \tilde{B} \subset \mathbb{R}^2$ which is an isometry between g and $\mathring{g} =<\,,\,>$.

Here some itemized hints:

(i) Let $\{E_1, E_2\}$ be any ON-frame for g, then $\nabla E_2 = E_1\omega_{12}$ with $d\omega_{12} = 0$. Show that there exists, at least locally on a small disk B around any point $p_0 \in U$, a change of frame $h\colon B \to \mathbf{SO}(2, \mathbb{R})$ so that the new ON-frame $(F_1, F_2) = (E_1, E_2)h$ satisfies $\nabla F_i = 0$.

(ii) Let $\{\sigma_1, \sigma_2\}$ be the dual frame to $\{F_1, F_2\}$, that is $\sigma_i \in \Omega^1(B, \mathbb{R})$ with $\sigma_i(F_j) = \delta_{ij}$. Verify that $d\sigma_i = 0$. You may want to use the formula for the exterior derivative from the last HW sheet, Problem 2. Now you should be able to construct $\varphi = (\varphi_1, \varphi_2)$.

Problem 2. Two Riemannian metrics g and $\tilde{g}$ on a manifold $U \subset \mathbb{R}^n$ are *conformally equivalent* if $\tilde{g} = e^{2u}g$ for a smooth function $u\colon U \to \mathbb{R}$. A Riemannian metric g is called *conformally flat*, if g is conformally equivalent to the standard Euclidean metric $\mathring{g} =<\,,\,>$. Examples of conformally flat metrics we know of are the hyperbolic and spherical metrics.

(i) Consider $U \subset \mathbb{R}^n$ with standard Euclidean metric $\mathring{g} =<\,,\,>$. What is the Levi-Civita connection for this metric, in other words, what is $\nabla_X Y$ for two vector fields X, Y? For $n = 2$ calculate the curvature function of this metric. This is just a sanity check.

(ii) Show that the curvature functions $K_{\tilde{g}}$ and K_g of conformally equivalent metrics in dimension 2 satisfy

$$K_{\tilde{g}} = e^{-2u}(K_g - \triangle_g u)$$

where $\triangle_g u$ is the Laplace operator of g (see next item). Calculating everything in an ON-frame usually is advisable.

(iii) Let (U, g) be a (local) Riemannian manifold (of any dimension). Verify that for $p \in M$ and any vector field $X\colon U \to \mathbb{R}^n$ the map $v \mapsto (\nabla_v X)(p)$ is a linear map between $\mathbb{R}^n$. Now convince yourself (provide an argument) that the trace of a linear map between $\mathbb{R}^n$ is well defined (independent of which basis you choose to present your linear map as a matrix). Now define the *divergence* $\operatorname{div} X$ of X to be the function

$$(\operatorname{div} X)(p) := \operatorname{trace}(v \mapsto (\nabla_v X)(p))$$

For a smooth function $f\colon U \to \mathbb{R}$ we have defined the gradient vector field $\operatorname{grad} f\colon U \to \mathbb{R}^n$ by $df = g(\operatorname{grad} f, -)$. We define the *Laplacian* of f with respect to the Riemannian metric g by

$$\triangle_g f := \operatorname{div} \operatorname{grad} f$$

Note that grad, and div via the Levi-Civita connection, depend on the Riemannian metric g. As a sanity check calculate the Laplacian for the standard Euclidean metric $\mathring{g} =< , >$ (and deja vu Calc III).

(iv) What is the curvature function K_g of a conformally flat metric $g = e^{2u}\mathring{g}$ in terms of u? Use this to calculate the curvature function of the upper half plane with the Poincare metric $g = \frac{1}{y^2}\mathring{g}$ and the spherical metric $g = \frac{4}{(1+|x|^2)^2}\mathring{g}$ on $\mathbb{R}^2$.

(v) Consider the strip $U = \{x \in \mathbb{R}^2 \,;\, 0 < x_2 < \pi\}$ with Riemannian metric $g = \frac{1}{\sin^2(x_2)}\mathring{g}$. What is its curvature function K_g?

Problem 3. Consider the energy functional

$$E(\gamma) = \frac{1}{2}\int_I g(\gamma'(t), \gamma'(t))dt$$

on curves $\gamma\colon I \to U$ into a Riemannian manifold (U, g). Show that E-critical curves (w.r.t. compactly supported variations) satisfy the geodesic equation

$$\nabla_{\gamma'}\gamma' = 0$$

Conclude in particular that E-critical curves have constant speed.

Problem 4. A surface of revolution is given by the following parameterization: let $R(s) \in \mathbf{SO}(3, \mathbb{R})$ denote the rotation around the z-axis of angle s and let $\gamma\colon I \to \mathbb{R}^3$, $\gamma(t) = (f(t), 0, h(t))$, be an arc length parameterized curve in the (x, z)-plane with $f(t) > 0$ for all $t \in I$. Consider the map

$$\varphi(s, t) = R(s)\gamma(t) \in \mathbb{R}^3$$

defined on $U := \mathbb{R} \times I$.

(i) Show that φ is a regular parameterization of a surface in $\mathbb{R}^3$ and calculate the induced Riemannian metric $g =< d\varphi, d\varphi >$.

(ii) Calculate the curvature function K_g of the Riemannian manifold (U, g).

(iii) Write down the ODE the curve γ has to satisfy in order for the curvature K_g to be constant. Can you find at least one solution in each of the cases $K_g = -1, 0, 1$? These would be surfaces of revolution of constant curvatures $-1, 0, 1$.

(iv) Calculate the equation for the geodesics (i.e. the energy critical curves) and show that the curves $s =$ *constant* are geodesics (longitudes).

(v) Which of the latitude curves $t =$ *constant* are geodesics?

Problem 5 (Bonus Problem: local Gauss-Bonnet)**.** Let $U \subset \mathbb{R}^2$ be diffeomorphic to a closed disk with smooth boundary given by a simply closed regular curve $\gamma\colon \mathbb{R} \to U$ (γ is periodic with period L). Let g be any Riemannian metric on U with curvature function K_g. Since γ is regular we may assume that γ is arclength parameterized $g(\gamma', \gamma') = 1$. Let $N\colon I \to \mathbb{R}^2$ be of unit length and normal to γ w.r.t. to the Riemannian metric g so that the ON-frame $\{\gamma', N\}$ along γ has the standard orientation. Convince yourself that $\nabla'_\gamma\gamma'$ is normal to γ (w.r.t. g). The *geodesic curvature* κ_g of γ then is defined by

$$\nabla_{\gamma'}\gamma' = \kappa_g N$$

Prove the local Gauss-Bonnet theorem

$$\int_U K_g\, dvol_g + \int_0^L \kappa_g dt = 2\pi$$

Hint: Stokes Theorem $\int_U d\alpha = \int_{\partial U}\alpha$ for $\alpha \in \Omega^1(U,\mathbb{R})$.
Double bonus question: if $\tilde{U}$ is diffeomorphic to a closed disk (with smooth boundary) and $U = \tilde{U} \setminus \cup_{i=1}^k D_i$ where $D_i \subset \tilde{U}$ are disjoint open disks, what would the Gauss-Bonnet formula for U be?

Remarks. Some of you might have noticed that in Problem 3 & 5 the expression $\nabla_{\gamma'}\gamma'$ is a bit mysterious. To understand the meaning of this expression will also make the calculations easier. In a certain way the next remarks are small additional exercises you should do. They isolate parts of local calculations you may have done yourself when working on the problem.
We have seen that on an n-dimesional Riemannian manifold (U,g) we have a unique torsion free metric connection ∇ which allows us to define the directional derivative of a vector field $X\colon U \to \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ at any point $p \in U$ via $(\nabla_v X)(p)$. The connection being metric gives us the "expected" product rule w.r.t. the inner product g_p on $\mathbb{R}^n$:

$$(d\, g(X,Y))_p(v) = g_p((\nabla_v X)(p), Y(p)) + g_p(X(p), (\nabla_v Y)(p))$$

for vector fields $X, Y\colon U \to \mathbb{R}^n$. Note here that the "usual" directional derivative $\mathring{\nabla}$ would not satisfy this equation (it satisfies it w.r.t. the standard Euclidean inner product $\mathring{g} =<\,,\,>$).
Now the usual directional derivative allows differentiation of maps $X\colon \tilde{U} \to \mathbb{R}^n$ where $\tilde{U} \subset \mathbb{R}^k$ for some k (not necessarily n), i.e. for $\tilde{v} \in \mathbb{R}^k$ we have $dX_q(\tilde{v})$, the directional derivative of X along $\tilde{v}$ at $q \in \tilde{U}$. In particular we know what $\frac{dX}{dt}$ is when $\tilde{U} \subset \mathbb{R}$ is an interval. To be able to do this w.r.t. to the Levi-Civita connection, we define the following:

Definition 1. Let $\tilde{U} \subset \mathbb{R}^k$, $U \subset \mathbb{R}^n$ and $f\colon \tilde{U} \to U$ be a smooth map. A *vector field along the map* f is a smooth map $X\colon \tilde{U} \to \mathbb{R}^n$ with the interpretation that for $q \in \tilde{U}$ we view $X(q)$ as a tangent vector at $f(q) \in U$.

Here the important examples :

(i) $\tilde{U} = U$ and $f = id_U$, then vector fields along f are just regular vector fields on U.
(ii) $\gamma\colon \tilde{U} \to U$ a curve (i.e. $k = 1$) then $X = \gamma'$ is a vector field along γ.
(iii) More generally, if $\tilde{X}\colon \tilde{U} \to \mathbb{R}^k$ a vector field on $\tilde{U}$, then $X = df(\tilde{X})$ is a vector field along f. The previous case is the special case when $k = 1$ and $\tilde{X} = 1\colon \tilde{U} \to \mathbb{R}$ is the constant vector field: $\gamma'(t) = d\gamma_t(1)$.
(iv) If $X\colon U \to \mathbb{R}^n$ is a vector field on U then $X \circ f\colon \tilde{U} \to \mathbb{R}^n$ is a vector field along f.

Our goal is to define the covariant directional derivative $(\nabla_{\tilde{v}} X)(q)$ for $q \in \tilde{U}$ and $\tilde{v} \in \mathbb{R}^k$ of a vector field X along f. A guiding principle should be that for a vector field $X\colon U \to \mathbb{R}^n$ we expect the property

$$(\nabla_{\tilde{v}}(X \circ f))(q) = (\nabla_{df_q(\tilde{v})}X)(f(q))$$

for the special vector field $X \circ f$ along f. Parse this formula for the standard derivative $\mathring{\nabla}$ and hopefully you can see that this is something you know well.
To avoid lots of sum symbols we will consistently use the notation $\underline{e}\,\xi := \sum_i e_i \xi_i$ for $\{e_i\}$ a basis of $\mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n)^t \in \mathbb{R}^n$.

Definition 2. Let (U, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $E_i\colon U \to \mathbb{R}^n$, $i = 1, \ldots, n$, be a frame with matrix valued connection 1-form $\omega \in \Omega^1(U, \mathbf{gl}(n, \mathbb{R}))$ defined via $\nabla \underline{E} = \underline{E}\,\omega$. For a vector field $X\colon \tilde{U} \to \mathbb{R}^n$ along f we can express $X = (\underline{E} \circ f)\,\xi$ with $\xi\colon \tilde{U} \to \mathbb{R}^n$. We define

$$(\nabla_{\tilde{v}}X)(q) := (\underline{E} \circ f)\,(d\xi_q(\tilde{v}) + \omega_{f(q)}(df_q(\tilde{v}))\xi(q))$$

for $q \in \tilde{U}$ and $\tilde{v} \in \mathbb{R}^k$ interpreted as a vector based at q. Dropping the dependency on q and $\tilde{v}$, we can write this as

$$\nabla X = (\underline{E} \circ f)\,(d\xi + (\omega \circ df)\xi)$$

At this stage contemplate what this definition would mean in the case $\omega \equiv 0$, which is the case for the standard derivative $\mathring{\nabla}$ and a constant frame $\{E_i\}$, e.g. an orthonormal basis of $\mathbb{R}^n$.

Now verify the following properties of this definition:

(i) ∇X is well-defined, i.e., if $\{F_i\}$ is another frame on U and $X = (\underline{F} \circ f)\eta$, then

$$(\underline{E} \circ f)\,(d\xi + (\omega \circ df)\xi) = (\underline{F} \circ f)\,(d\eta + (\omega \circ df)\eta)$$

(ii) If $X\colon U \to \mathbb{R}^n$ is a vector field on U then our guiding principle formula above holds for the special vector field $X \circ f$, that is

$$(\nabla_{\tilde{v}}(X \circ f))(q) = (\nabla_{df_q(\tilde{v})}X)(f(q))$$

(iii) The metric property for the Levi-Civita connection is reflected by the following formula for vector fields X, Y along f, $q \in \tilde{U}$ and $\tilde{v} \in \mathbb{R}^k$:

$$(d\,g(X,Y))_q(\tilde{v}) = g_{f(q)}((\nabla_{df_q(\tilde{v})}X)(q), Y) + g_{f(q)}(X, (\nabla_{df_q(\tilde{v})}, Y)(q))$$

(iv) The torsion free property for the Levi-Civita connection is reflected by the following formula: let $\tilde{X}, \tilde{Y}$ be vector fields on $\tilde{U}$, then the special vector fields $df(\tilde{X}), df(\tilde{Y})$ along f satisfy

$$\nabla_{\tilde{X}}df(\tilde{Y}) - \nabla_{\tilde{Y}}df(\tilde{X}) = df([\tilde{X}, \tilde{Y}])$$

Using those properties Problem 3 becomes quite easy to solve. Also, the interpretation of the expression $\nabla_{\gamma'}\gamma'$ should now be clear, even though it is not consistent with the notation just introduced.