Advanced Calculus Midterm
due 5/5/17

Show all of your work and give full explanations. Write legibly and hand back your exam stapled.

Problem 1. Which of the vector fields F below is gradient field? If yes, provide a potential function f such that $F = \operatorname{grad} f$.

(i) $U = \{x_1, x_2)\,;\, x_1 > 0\}$ and $F\colon U \to \mathbb{R}^2$ is given by
$$F(x_1, x_2) = (1/x + 2xy, x^2 + y^2)$$

(ii) $U = \mathbb{R}^2 \setminus \{0\}$ and $F\colon U \to \mathbb{R}^2$ is given by
$$F(x_1, x_2) = \frac{1}{x_1^2 + x_2^2}(x_1(x_1^2 + x_2^2 + 1), x_2(x_1^2 + x_2^2 - 1))$$

(iii) $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$F(x_1, x_2) = (x_1^2 - x_2^2 + x_1, 2x_1x_2 - x_2)$$

Problem 2. Consider the "wormhole" $M \subset \mathbb{R}^4$ in 4 dimensions given by the equation
$$x_2^2 + x_3^2 + 2x_4^2 = (\cosh x_1)^2$$

(i) Find all the points on the wormhole closest to the origin.

(ii) Verify that the tangent 3-planes $x + T_xM$ to the wormhole at those closest points $x \in M$ are perpendicular to x.

Problem 3. Newton's equation for the position $x(t) \in \mathbb{R}$ of an oscillating spring with non-linear friction and restoring force term is assumed to be
$$x'' + a(x')^3 + b(x^2 - 1) = 0$$
where $a, b > 0$ are constants.

(i) Rewrite Newton's 2nd order ODE into a first order ODE of the form
$$(x, v)' = F(x, v)$$
where $v = x'$.

(ii) Find all the zeros (equilibria) of the vector field $F(x, v)$.

(iii) Linearize the first order ODE at the equilibria and determine (for each equilibrium point) whether the linear ODE has a sink, source, saddle or spiraling point.

(iv) Make a heuristic statement about the behavior of the solutions of the non-linear ODE near the equilibrium points.

Problem 4. Consider the function
$$f\colon \mathbb{R}^2 \to \mathbb{R} \qquad f(x) = (x_1^2 - 1)^2 - x_2^2$$

(i) Determine all the critical points $\mathring{x}$ and critical values $f(\mathring{x})$ of f.

(ii) Characterize the critical points as local maxima/minima/saddle points.

(iii) Now restrict the domain of f to the closed square
$$R = \{(x_1, x_2)\,;\, |x_1| \le 1, |x_2| \le 1\}$$
of side length 2 centered at the origin. Determine the global maxima and minima of f on R.