**132H Final Exam**

Due by Dec 18, 2018, at noon

You can only use material covered in class. You need to show all your work and quote the relevant results, tests, etc from class if you use them. For instance, if you apply the ratio test you should say "from the ratio test we can now conclude that etc". You can use calculators for numerical calculations only, not for symbolic computations. You can work in groups, but everyone has to write up their solutions according to their own style and understanding. Copying someone else's work (even if slightly altered) can easily be spotted and is in violation of the honor code with consequences for all parties involved. Same holds if you try to outsource your work. All problems can be done with the material covered in class.

**Problem 1.** Show that the infinite series, called the *Riemann zeta function*,

$$\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{n^z}$$

converges absolutely for any complex number $z \in \mathbb{C}$ for which $\mathrm{Re}(z) > 1$.

**Problem 2.** Calculate all $n$ zeros $z_1, \ldots, z_n \in \mathbb{C}$ of the degree $n$ polynomial $P(z) = z^n - 1$ and show that the points $z_1, \ldots, z_n \in \mathbb{C}$ form a regular $n$-gon. *Hint:* $1 = e^{2\pi i}$.

**Problem 3.** This problem has three parts. The first question is about a finite degree polynomial. The second question concerns an infinite degree polynomial, that is, a power series. The third question calculates the value of an infinite series.

(i) Let $x_1, \ldots, x_n \in \mathbb{R}$ be non-zero numbers. Verify that

$$P(x) := (1 - \frac{x}{x_1}) \cdots (1 - \frac{x}{x_n})$$

is a degree $n$ polynomial with zeros at $x_1, \ldots, x_n$ of the form

$$P(x) = a_n x^n + a_{n+1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0 = 1$ and $a_1 = -\sum_{k=1}^{n} \frac{1}{x_k}$. Calculate also $a_2$ and $a_n$.

(ii) Now consider the infinite polynomial

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots$$

which is just the power series expansion of $\sin x$ divided by $x$. In analogy to the finite degree case (i) write

$$P(x) = (1 - \frac{x}{x_1})(1 - \frac{x}{x_2})(1 - \frac{x}{x_3}) \cdots$$

as an infinite product where $x_k$ are the infinitely many zeros (what are they?) of $\frac{\sin x}{x}$.

(iii) Use the ideas of (i) and (ii) (and a renaming of $x^2 = u$, that's a hint...) to show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

This is an accomplishment due to Euler. To this day there are no known formulas for the sums of the odd power reciprocals of the integers, e.g. $\sum_{k=1}^{\infty} \frac{1}{n^3}$.

**Problem 4.** Show the formula

$$\int_0^{\infty} x^n e^{-x}\, dx = n!$$

for $n \geq 1$.

**Problem 5.** Consider the differential equation $f''(x) + f(x) = 0$ for a function $f(x)$. Assuming that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ has a power series expansion calculate the coefficients $a_k$ assuming the initial conditions $f(0) = 1$ and $f'(0) = 0$. Can you identify/recognize the function your power series describes?

This differential equation describes the up and down motion (where $f(x)$ is the elongation) as a function of time $x$ of a mass attached to the end of a spring suspended from the ceiling. The initial conditions posed mean that the mass is pulled down by one unit from its equilibrium, and then let go (without any initial push). You can carry out this experiment by getting a not too stiff spring from a hardware store, hook it to the ceiling, and hook a weight to the lower end, pull the weight down and release it, and stare at it...