Great problems of geometry and space

28 May 1983

Summary: *To do mathematics is to raise great mathematical problems, and try to solve them. Eventually to solve them. This time, we shall treat problems of geometry and space, and we shall classify geometric objects in dimensions 2 and 3. Dimension 2 is classical: it's the classification of surfaces, which are obtained by attaching handles on spheres. One can also describe surfaces by using the Poincaré-Lobatchevsky upper half plane. What happens in higher dimensions? In dimension ≧ 5, Smale obtained decisive results in 1960. Last year, Thurston published great results in dimension 3. He conjectured the way such objects can be constructed starting with simple models, and also how one could obtain them from the analogue of the upper half plane in 3 dimensions. He proved a good part of his conjectures. We shall describe Thurston's vision.*

First Part: Rubber geometry.
Curves, surfaces, equivalences, octopusses, sums of geometric object.

Second Part: The geometry of distances.
Euclidean geometry, non-euclidean, distances, motions, translations, rotations, symmetries, identifications.

The link between the two and Thurston's conjecture.

It is uncommon on a Saturday afternoon in May to see 230 persons come, not only to listen to a conference on mathematics, but also to participate, answer questions, in short think about mathematics and get pleasure out of it. To be sure, the enthusiasm of the lecturer, the energy which comes from him, and the care which he exercises to explain his subject and ideas can hardly leave an audience insensitive. On the other hand, it seems clear that the pleasure is shared. First of all by me, but also by Serge Lang. One sees in him what should be natural for any good teacher, satisfaction in the face of positive reactions by the public, and the relevance of the questions which come from his audience, especially by some high school students. After the success of the first two conferences, one can easily understand that I wanted to invite him again, and that he accepted, not without some hesitations because he said that it would be difficult to choose a genuine mathematical topic which would nevertheless be understandable by a broad audience. Two weeks later, he phoned me from Germany to tell me that he had found a possible geometric subject, but that he would have to learn it. To my question: "In which books?" he answered: "I don't know how to read . . . Or rather, I know how to read but I don't like it. In a book, not everything is of equal importance, but one doesn't know it until one has read everything. It goes much faster to ask a friend to explain this stuff. It's more lively, and I can ask questions."

During the course of the year, I then received successive versions of his talk, which testified to his concern for clarity and simplicity. But it is hardly necessary for me to say here that these versions were only pale sketches compared to the following text, which reproduces faithfully the tape recording of his marathon talk, which lasted over three hours.

J.B.

The conference

The first hour

This is the third time that I come here, to the Palais de la Découverte, to do mathematics with you. Mr. Brette invited me the first time, and it worked, so I came back.

I see quite a few people here who came last time, How many were here last year?

[*About fifty hands go up.*]

Good. I see Antoine over there, he's already been here twice, so he's quite faithful. Those who were here last year perhaps remember that just before starting the conference, I had looked at some high school book in Brette's office, and I had become quite upset because it was so lousy. It took me a good twenty minutes to get over it. I don't know if you noticed, today, before the conference, you heard a record of lute music, which is the music I like best. Brette put it on to calm me down. [*Laughter.*]

Two years ago, I did something on prime numbers, and last year, I did what's here on this reprint, what's called diophantine equations. And I asked people what mathematics meant to them. One lady told me: "It's to work with numbers." Well, those answers are for the birds, because this is not at all what it means to do mathematics. I wanted to show you what mathematics are about, what the great problems of mathematics are about, and why one gets excited about them.

Actually, in the first two talks, I did things which were related a little to algebra, and even a lot. In particular, last year, I wrote down some formulas, and then six persons immediately walked out, because formulas . . . well, people don't like them so much. But sometimes formulas are necessary. Still, I wondered whether it would be possible to do something without any formulas, without any connection to algebra, and without numbers. This means doing something geometric, in space, on problems having to do with geometric objects.

That's not the mathematics I usually do myself. Personally, I lean toward algebra and number theory. So I thought about it, as I left Paris for Bonn, in Germany, trying to figure out what I could talk about this year. I go to Bonn like this every year, for the last twenty, twenty-five years. Hirzebruch organizes a conference, and the people who go there are mostly interested in geometric subjects. I talked to some of them, and I realized that it would be possible to do one topic, on some recent research, discovered about a year ago.

It's very nice in Bonn. Mathematicians try to hold their conferences in pleasant surroundings, and there, we do mathematics between a glass of Rhine wine and a strawberry tart—strawberries are in season just at that time—and also a boat trip on the Rhine.

There is still some time left to do mathematics, and that's where I learned the subject I am going to talk about today, some recent discoveries of a guy named Thurston. I learned it from Walter Neumann. We spent three hours in front of a blackboard, and he taught me what Thurston had done. And today, I pass it on to you.

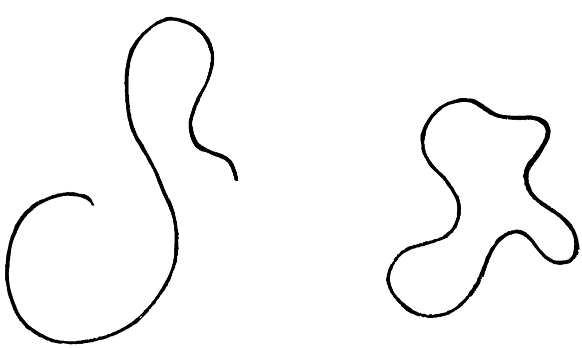
I really don't know if I'll be able to go on like this, finding other subjects, because it isn't so easy. It must be real mathematics, done by real mathematicians. On the other hand, I must be able to explain it to a Saturday afternoon audience. And also, as in all aesthetic situations, somebody may like it, and somebody else may not like it. So it may not work out. It's a question of personal tastes, and the personal reactions which you may have toward any specific topic.

All right, so what I want to do, is to classify geometric objects. We are immediately motivated for this. After all, we live in a space with at least three dimensions, like this. But you all know that there may be more than three dimensions. So we want to describe the kind of space we live in, we want to know what it looks like. Locally, for instance in this room, it's a three-dimensional space.

As a model, it's OK for this room. But we already know that if you look very far out, it doesn't work. We know that the euclidean model is wrong. It works in restricted cases, but it does not apply in other situations. So what do physicists do? They try to find out which models are applicable. But a mathematician, that is, a pure mathematician, doesn't care whether the models he thinks about can be applied or not. He constructs nice models, geometric models, and if they are beautiful, that's what matters to him. He doesn't care whether these models can be used to describe the universe or not.

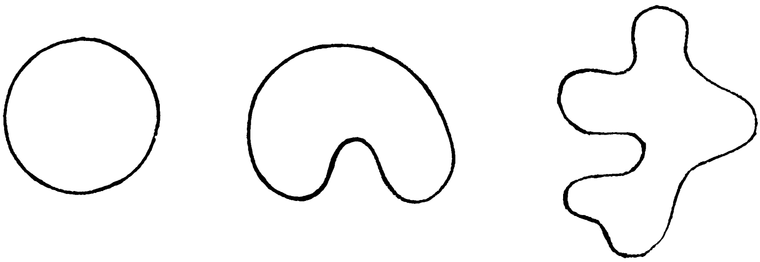
And we can do such geometric models in dimension 1, in dimension 2, in dimension 3, or also higher dimensions, like 4, 5, or whatever. I thought for a moment that I could do something today in higher dimensions, but I realized rapidly that I could not, at least in an hour and a half. It would have taken too much preparation. So I'll limit myself to dimensions 1, 2 and 3.

So, one-dimensional objects are like this, they are curves.



A LADY. What about a straight line?

SERGE LANG. A straight line is a special type of curve. Now if I take a circle, and other curves like that, they look like each other.



Should we consider them to be equivalent? What common properties do they have?

SOMEONE. They are closed.

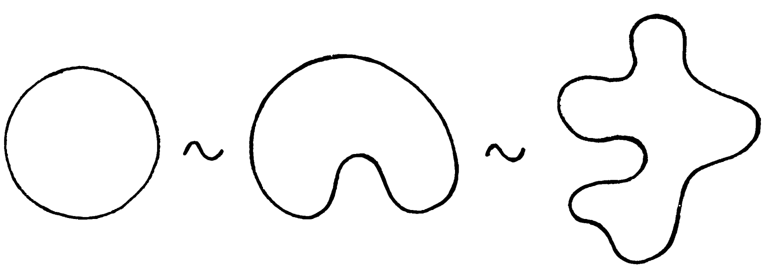
SERGE LANG. Yes, they are closed, they turn around. If I take just an interval, like this:

then it doesn't turn around, it's an interval.

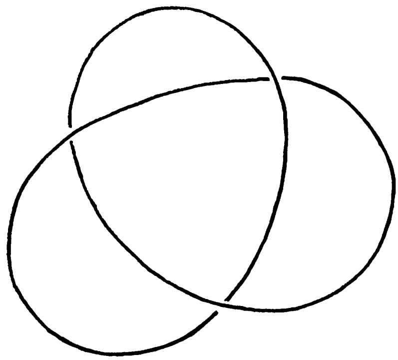
But the three curves are closed. For many properties, we don't want to distinguish between these three curves. So we say these curves are equivalent.

What does "equivalent" mean, in general? Well, I don't want to give a formal definition, but I can say informally that we suppose everything is made out of rubber. We are dealing with rubber geometry. We say that two objects are equivalent if by pulling in one direction, pushing in another, if these objects were made out of rubber, then I could deform one into the other. This gives me a notion of equivalence.

So if the curve is a rubber band, it's clear that I can deform it to the other curve, or that I can deform it into a circle. So these curves are equivalent. I use the sign $\sim$ to denote equivalence. So I can write:



I can also draw something like this:



Is this thing closed or open?

THE AUDIENCE. It's closed.

SERGE LANG. Then is it equivalent to the others or not? Suppose it's a rubber band. Who says it's equivalent to the others?

[*Some hands go up.*]

Who says it is not equivalent?

[*Other hands go up.*]

Who keeps a prudent silence? [*Laughter.*] You, for instance. [*Serge Lang points to a lady in the third row.*]

THE LADY. It's not equivalent. There is a knot.

SERGE LANG. Yes, there is a knot. When I said that the other curves are equivalent, I could deform them into each other in the plane. I mean, if they were rubber bands, I could make the deformation entirely in the plane. But the knot, over there, it lives in three-dimensional space, and your intuition is right: I cannot deform it into the circle in three-dimentional space. In some sense, the knot is therefore different from the circle, and from the other curves. However, can you conceive a situation when I could deform the knot into a circle? Antoine, what do you say.

ANTOINE. [*The answer cannot be heard on the tape.*]

A LADY. Sometimes you can make two knots which are opposite to each other, and undo each other.

SERGE LANG. For now, the knot is in 3-dimensional space. But there is no reason to limit ourselves to this space. It is true that in four-dimensional space, we can deform the knot so that it becomes a circle. One can also prove that this is impossible in 3-space. Although we can rely on our intuition in 3-dimensional space, when proving things in higher dimensions, one should first write things down more rigorously, and our intuition becomes rather delicate. I also want to make you understand that things aren't that simple.

We now see that we can raise two different questions:

Can we deform the knot into the circle in 3-dimensional space?

Can we deform it abstractly, or in a higher dimensional space?

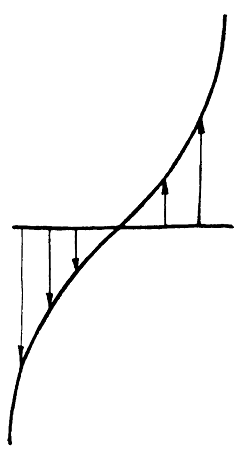
The answers are different, depending on the space in which we embed the knot.

At first, I didn't say where you could make the deformation, when I defined the notion of equivalence. Now I say that I allow deformations in spaces of arbitrarily high dimension, bigger than 2 or 3. So the dimension of the circle, which is 1, has to be regarded as entirely different from the dimension of the space in which we consider the circle.

Now I want to say something else about deformations. Take something which is not a circle, say an interval, like this, with or without its ends.

with the ends without the ends

If I include the ends, then I say that the interval is closed. If I don't include the ends, then I say that the interval is open. Suppose the interval is made of rubber, and I deform it, like this. [*Serge Lang draws as he speaks.*]



The points to the right, I move them up and the points to the left, I move them down. So I take the rubber band, and stretch it up as I go to the right, but faster and faster. And when I go left, I stretch it down, also faster and faster. Then we see that the interval is equivalent to a curve which goes arbitrarily far away, which extends to infinity as one sometimes says.

[*Someone raises their hand.*]

SERGE LANG. Yes?

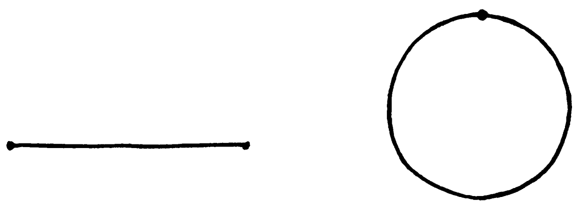
A LADY. It's going to close up at infinity.

SERGE LANG. No, infinity is not a point. Take a line, like this:

This line does not close up.

A GENTLEMAN. If it's made of rubber, one can close it up. [*Laughter.*] It is not an interval, but with an interval, you can also make up a circle.

SERGE LANG. Watch out! If you close up the line, or if you close up the interval, then you have to put some points at the end over each other. Take an interval containing its end points. If I join up the end points, then I do get a circle.



THE GENTLEMAN. But the interval can be deformed into a circle.

SERGE LANG. No, because if I deform it into a circle and I identify the two end points, then I do something which I don't want to allow in the definition of a deformation. I want to use the word in the sense that I do not allow identifications. If two points are distinct, then they must remain distinct during the deformation.

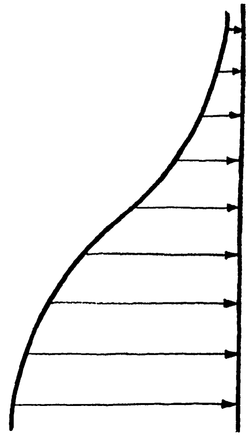
GENTLEMAN. But if you juxtapose them . . .

SERGE LANG. No, no. I don't want to! [*Laughter.*] It's a question of definitions. For the applications which I want to make here, I want to use the word "deformation" to mean that if two points are different, then they must remain different under the deformation. OK?

GENTLEMAN. Yes.

SERGE LANG. Good. Of course, there are other notions where identifications are allowed. In fact, in a short while, I shall discuss such notions and how to use them. But here, for deformations, I don't allow it.

I just wanted to show you this specific phenomenon, that I can deform an interval without its end points into an infinite band, which is itself equivalent to an infinite line. I can draw an equivalence between this infinite thing and the infinite line like this:

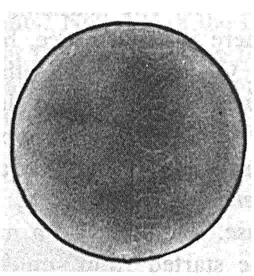


So I can straighten out the curve to a line. And that's the notion of equivalence that I want to work with.

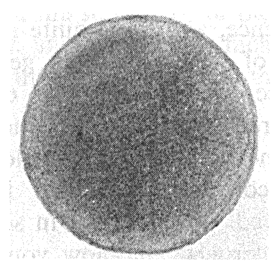
All right, we've been talking about things of dimension 1. But already in dimension 1, we see that we can raise some problems. You might think that everything is known, but that is not the case.

Next, we look at dimension 2. It's going to get a little more involved. Objects of dimension 1 are called curves. Objects of dimension 2 are called surfaces. And there are surfaces with boundary and surfaces without boundary.

For an example of a surface, take the disc, the interior of the circle. If I put the circle together with its interior, then I get a surface with boundary. The circle is the boundary of the disc. So we can consider the disc as a surface without boundary if we leave out the circle, and with boundary if we include the circle.

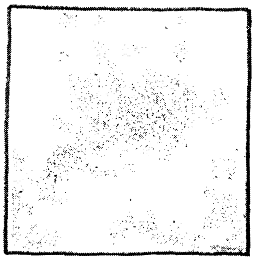


with boundary



without boundary

Now, if the disc is made of rubber, then I can represent it in other ways:

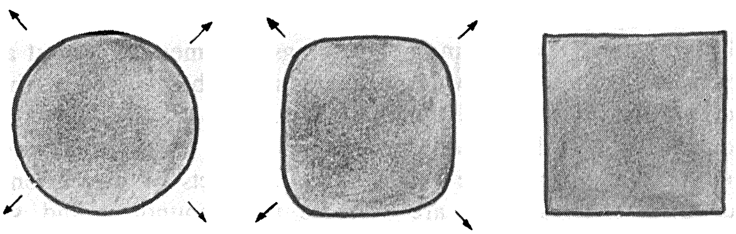


for instance, I can take the interior of a square. The boundary is then the perimeter of the square.

If everything is made out of rubber, are they equivalent?

THE AUDIENCE. Yes.

SERGE LANG. That's right, they are equivalent, I can take the disc and stretch it out to obtain a square, the interior of a square.



And the boundary of the disc, that is the circle, will become the boundary of the square.

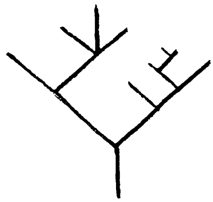
[*A hand goes up.*]

SERGE LANG. Yes?

GENTLEMAN. But there is some difference, because of the derivatives.

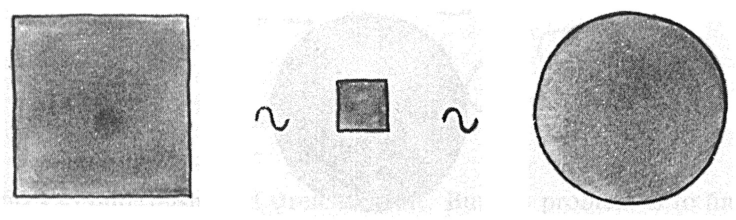
SERGE LANG. Of course, there are corners. The gentleman says there is a difference, and he is quite right. There is a difference, but not from the point of view of rubber geometry. One can define other kinds of equivalence, for which the two objects would not be equivalent, because when I stretch out the disc and created a corner, then obviously this corner is not smooth. You could even say that from a certain point of view, the corner is disgusting. [*Laughter.*] It's not smooth, and it's not nicely curved. It's different in some sense. There is also a mathematical theory of corners, and now you see, we started from something rather simple, and already we can ask a lot of questions, which develop like a tree:

A person who walked out after about twenty minutes told the guardian: "I don't know if it's me who is not smart enough, but all this is just a farce."

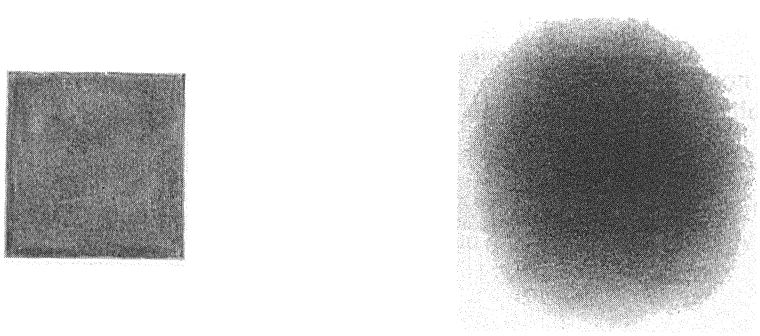


We climb up the tree, and we find two, or even several possibilities to go on. Depending on which equivalence relation you work with, you will find different answers for the same question. But right now, I only want to consider the rubber equivalence. Then the disc and the square are equivalent.

Of course, this does not depend on their size. I can make the square big or small. If it's made out of rubber, it will still be equivalent to the disc.



If I take just the interior of the disc, without the circle, then I get a surface without boundary, or the square without boundary. This is similar to the interval, without its end points. You remember the interval without the end points? Now I take the interior of the square, or of the disc which is equivalent to it, without the boundary, and the plane which extends to infinity in all directions. Do you think that the interior of the square is equivalent to the plane?



Who says yes?

A GENTLEMAN. The plane is indefinite?

SERGE LANG. It's the plane, yes, it's infinite.

GENTLEMAN. The square is without boundary?

SERGE LANG. Right, it does not have a boundary. I took it out, that's why I drew the dotted lines.

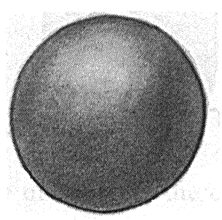
GENTLEMAN. Then it's also indefinite?

SERGE LANG. Like you say, it's indefinite.

GENTLEMAN. Then they are equivalent.

SERGE LANG. That's right. The square without boundary is equivalent to the plane. To summarize, every interval without boundary is equivalent to an infinite straight line, and every square or disc without boundary is equivalent to the whole plane.

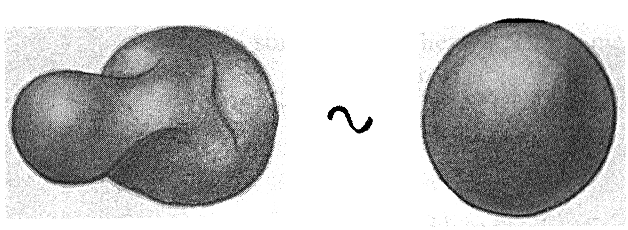
But please note: if I take the square without boundary, I can still add the boundary if I want. Suppose however that I take a sphere, like this, the surface of a sphere:



It's a surface, but it does not have a boundary, OK? And if I stretch it, is it possible to stretch it in such a way that parts of it go as far away as you want?

GENTLEMAN. You can blow up the balloon indefinitely.

SERGE LANG. Watch out! I don't want to tear up the balloon. [*Laughter.*] The objects have to remain equivalent. I blow up the balloon, and punch it in or out some, like rubber, but I am not allowed to tear it up.



But if I stretch it, would it be possible to do what I did with the interval without its end points, to send parts of it as far away as you want? Who says yes?

[*Some hands go up.*]

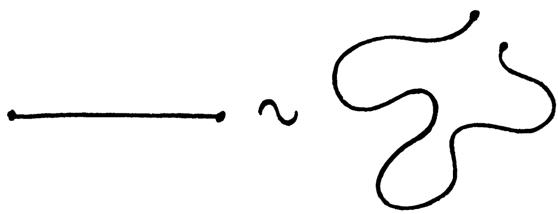
Who says no? Actually, the answer is no. For instance, if I take the interval with its end points, would it be possible to stretch it so that it becomes equivalent to something infinite?

A LADY. It would be bounded.

SERGE LANG. That's right, one can prove that it is impossible. The interval with its end points is not equivalent to an infinite object.

LADY. Do you mean that the end points are fixed?

SERGE LANG. Oh no! They are not necessarily fixed, you can move them. For instance, it's equivalent to this thing here.



It suffices to pull, push, and stretch a little. But the problem is to find out if I can stretch faster and faster, as I did for the interval before. What happens if I stretch faster and faster, is that the end points have nowhere to go. Before, the points of the interval which came closer and closer to the extremities went higher and higher; or lower and lower. So to include the extremities in the deformation, I would need to tear off the end points. And I don't allow that.

A GENTLEMAN. You can put the end points at infinity.

SERGE LANG. No, we have to remain in the plane, there is no point at infinity in the plane, there are just points which go out as far as you want, it's not the same thing.

GENTLEMAN. Why is it forbidden?

SERGE LANG. It's forbidden in order to define the notion of equivalence. It's not forbidden in principle, it's not absolutely forbidden. You can add a point at infinity to the plane for other applications, but not for those I want to make today.

So you have to distinguish between things which have the property that under some deformations, some parts of them can be sent arbitrarily far away, and things which do not have this property. So let me write down a definition.

> I say that something is compact if it contains its boundary (whenever the boundary exists), and if no deformation of this thing extends arbitrarily far away. In other words, if every deformation of this thing is bounded.

All of this is to come to the point of saying that the sphere is compact. Of course, the three-dimensional space in which we live goes to infinity . . . [*hesitating*], at any rate the naive model that we have in mind goes to infinity. But suppose you live on a sphere, and that you are very, very small. When you look around you, in any direction, it looks like a plane . . .

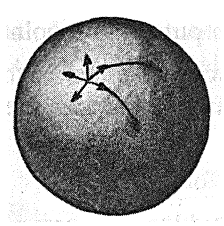
[*A hand goes up.*]

SERGE LANG. Yes?

A COLLEGE STUDENT. But the sphere is without boundary. You said: "compact without boundary".

SERGE LANG. Ah! if the surface has no boundary, it means that it contains its boundary. The terminology must accept this way of expressing yourself. If something doesn't have a boundary, then it can't help but contain its boundary, because there isn't any. [*Laughter.*] You must allow this possibility, because otherwise, you'll have a very hard time making simple mathematical statements.

Let's go back to people living on a sphere. Maybe they will see only a plane, even with good telescopes, and they will quickly come to the conclusion that the space on which they live is a plane. But suppose that a thousand years later, they make better telescopes, then maybe they will discover some curvature, they will see that space is curved, and they can start asking questions.



This is precisely what happened until Columbus. People thought everything was flat, except clever people, but there weren't so many of those.

THE AUDIENCE. So what's new! [*Laughter.*]

SERGE LANG. OK, so we look like that, and we can ask what happens if we keep on going, whether we can come back where we started from, or whether we go to infinity. The sphere is an example of something which is compact. If you start from some point, and keep going straight ahead of you in a given direction, then you come back where you started from.

Can you give me examples of other surfaces like that, compact surfaces?

THE AUDIENCE. A cube.

SERGE LANG. Yes, the surface of a cube, but it is equivalent to a sphere. Give me an example which is not equivalent to a sphere.

A GENTLEMAN. A torus.

SERGE LANG. What?

A GENTLEMAN. A torus.

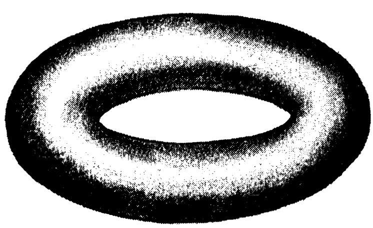
SOMEONE ELSE. You make a hole in the sphere.

SERGE LANG. Are you a mathematician?

THE GENTLEMAN. A little.

SERGE LANG. That's already too much! I would like mathematicians not to intervene, because otherwise, it's cheating. [*Laughter.*] Of course, mathematicians know the answer, but I am not giving this conference for them. [*Serge Lang throws the chalk at the gentleman. Laughter.*]

So, you dig a hole, and you find this object, which has a hole in the middle.



Then one can show that this surface is not equivalent to the sphere, because of the hole. Now can you give me an example of a surface which is not equivalent either to the sphere or to the torus?

SOMEONE. A Klein bottle.

SERGE LANG. Some of you know too much.[1]

A CHILD. A pyramid?

[1] I don't want to go into this kind of technicality at this point.

SERGE LANG. No, that's equivalent to the sphere.[2]

A LADY. A box without its top?

SERGE LANG. Yes, but it will have a boundary. I want a surface without boundary. The preceding one doesn't have one, the sphere neither. I want a compact surface.

A COLLEGE STUDENT. You can make two holes, like eyeglasses.

SERGE LANG. There you are, that's what I wanted you to say. But are you a mathematician?

THE STUDENT. Yes.

SERGE LANG. Oh no, no, don't do this! Naturally, if you are a mathematician, you'll say: make two holes. But you are not playing the game. [*Laughter.*] That's why I am asking you not to intervene. I want to make people think for themselves.

So, you are right, I can make two holes. Like this.

And if I want still another example, what do I do?

A STUDENT. A torus with a knot.

A LADY. You can continue to make more and more holes.

SERGE LANG. Very good. You were here the first time madam? Two years ago? You don't remember? I remember you very well. Anyhow, you can make more and more holes. And there is no limit to the number of holes, except that there can only be a finite number.

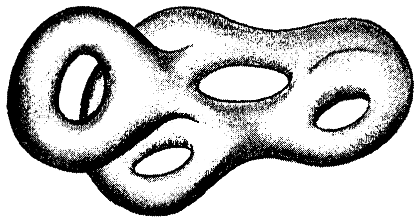
---

[2] The audience has all kinds of people, including twelve year old children, high school and college students, engineers, and retired people. I learned subsequently that this child is 12 years old. Her teacher asked some students in her class who attended the conference to write up their impressions afterwards. This one wrote:

> Of course, sometimes, I was a little confused, as when Mr. Lang asked for an example different from the sphere in dimension 2. I answered: "A pyramid", because I understood that Mr. Lang asked for a similar example. Otherwise, everything went well.

Another one said:

> If I knew that what I say would be written down, I would have raised my hand more often.



So that's a theorem:

> Compact surfaces, without boundary, are completely characterized, up to equivalence, by the number of holes. And there are no others.

I also have to make an additional hypothesis in the statement of the theorem. I should have said: an orientable surface. But I don't want to get into this kind of question now. So forget I said it. If I had not said it, then someone would have raised a fuss, some mathematician. [*Laughter.*]

[*As Serge Lang writes the statement of the theorem on the blackboard, space runs out, which forces him to erase another part of the blackboard.*]

So, there is no space left! Then you must all write to the Secretary of Education, so that he gives more funds to Mr. Brette for the Palais de la Découverte, so he can get more blackboards, and bigger ones, in a big room, and so on . . . Non-compact funds, if possible. [*Laughter.*] You all write to the Secretary, after the conference. I write the theorem:

> Surfaces which are compact, without boundary and orientable (just for my conscience) are characterized up to equivalence by the number of holes.

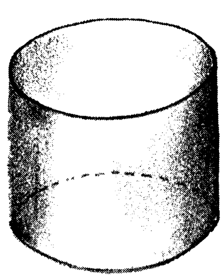
That's the general model for surfaces.

Now let's look at surfaces with boundary.

A GENTLEMAN. And when there are only holes left?

SERGE LANG. There always remains some surface. I am doing all this as an introduction to objects in three dimensions, when it's going to become much more serious.

OK, now I draw a surface with boundary. Someone had already mentioned a cylinder.



What's the boundary of the cylinder?

A GENTLEMAN. A circle.

SERGE LANG. Right, there is a circle on top and a circle on the bottom. The boundary of the cylinder is composed of two circles.

A LADY. There are also some edges.

SERGE LANG. No, because when the cylinder turns around, and you look at it sideways, you won't see these edges.

LADY. Then there are two boundaries?

SERGE LANG. Yes, or rather, there is a single boundary composed of two circles. Nobody said that the boundary has to consist of only one piece. It doesn't have to be connected.

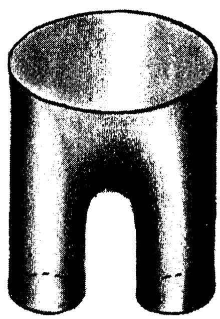
Now I'll draw one which is a little more fun. Who can tell me how to draw a surface with a boundary consisting of more than two pieces?

GENTLEMAN. A face.

SERGE LANG. Yes, for instance.

LADY. A sieve. [*Laughter.*]

SERGE LANG. Yes, very good. Let me draw another one.



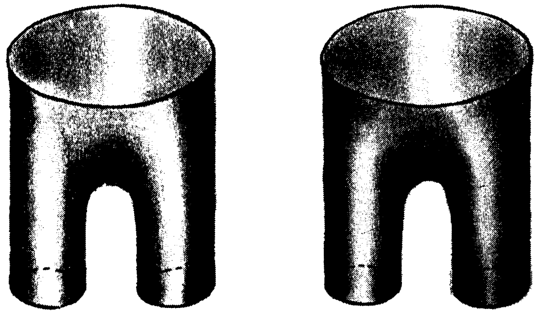
What is this?

GENTLEMAN. An upside down vase.

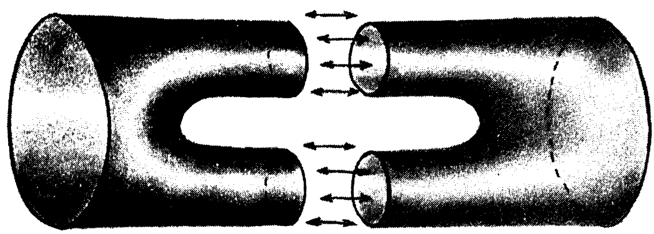
OTHERS IN THE AUDIENCE. A pair of pants.

SERGE LANG. Yes, a pair of pants. The boundary consists of the circle on top, and the two circles on the bottom. So the boundary has three pieces.

Now I am going to do something that mathematicians like a lot. Mathematicians like to combine things to make sums. Suppose you have two pairs of pants.



What can I do to them? If I take a circle on each one of them, I can glue them together.



And I can do the same thing with the other leg. Then I obtain something which I can call a sum of the two pants.

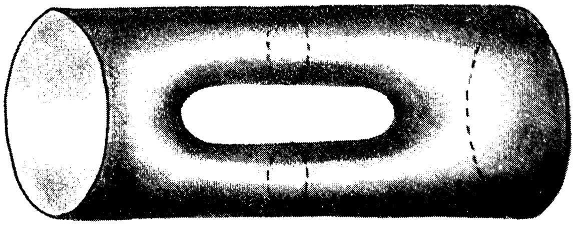
GENTLEMAN. But you don't have the right to do this, you are identifying things.

SERGE LANG. Now I have the right. I am doing sums, I am sewing the pants together. [*Laughter.*] I have the right to sew. We are coming to the point when I have the right to identify.

LADY. Why didn't you have the right to identify a while ago, but now you have the right to identify?

SERGE LANG. You always have the right to identify, to put two points together. Do what you want. But for what purpose? To define the notion of equivalence, you don't have the right. I did not say that when I identify, then I obtain an equivalence. I said I obtain a sum. It's not the same thing. For equivalences, I am not allowed to identify points. For sums, I do have the right. To do a sum consists in identifying pieces of the boundary.

So if I draw the sum, I obtain something like this, with a hole, and still a boundary which contains two circles.

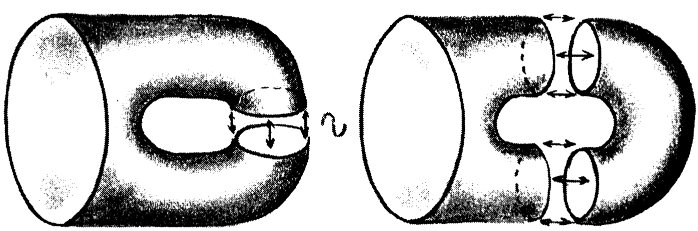


Now I can again do a sum, to eliminate the circles.

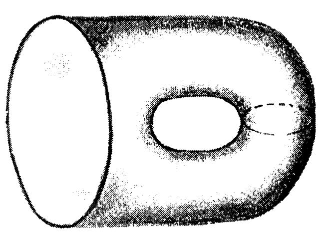
GENTLEMAN. In any case, you identify the boundaries of two distinct things.

SERGE LANG. Yes, two distinct objects. I take two pairs of pants, yours and mine, and I sew them together. [*Laughter.*]

Now I could also make a sum by taking a single pair of pants, and by identifying the two pieces at the boundary of the legs. It also gives me a surface with a hole.



It's the same thing as if I had taken the sum of the surface with a cylinder, it's equivalent. I would get a hole, but there remains a single circle as the boundary.



Now this circle, I want to eliminate it. How do I do that? What sum do I have to take to eliminate the boundary completely?

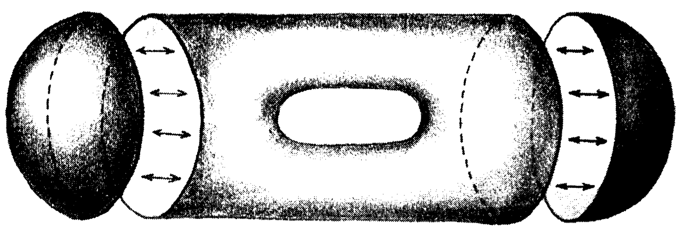
GENTLEMAN. A half-sphere?

LADY. A cover.

SERGE LANG. Precisely. A cover, which is a disc with its boundary.

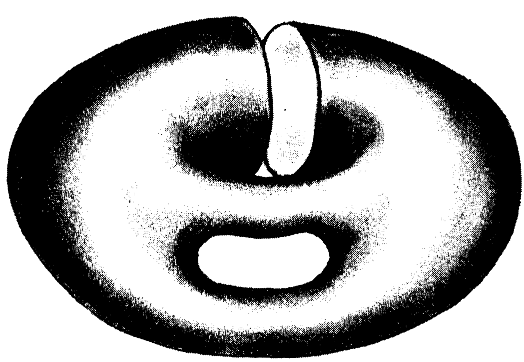
You glue them together, and you get something without boundary. That's what I wanted to show you. If I take surfaces with boundaries, with circles as their boundaries, and I take their sum, I can add them a certain number of times, and get a surface with holes, but without boundary.

Take the pants again. I can eliminate the boundary first by sewing the bottoms together, and then putting a cap at each end. I get a torus.



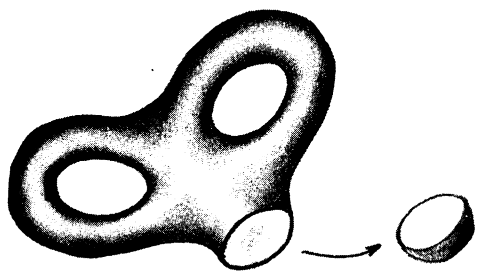
A STUDENT. But you could also have sewn them together along the belts.

SERGE LANG. Yes, I could, but that would have created a new hole.

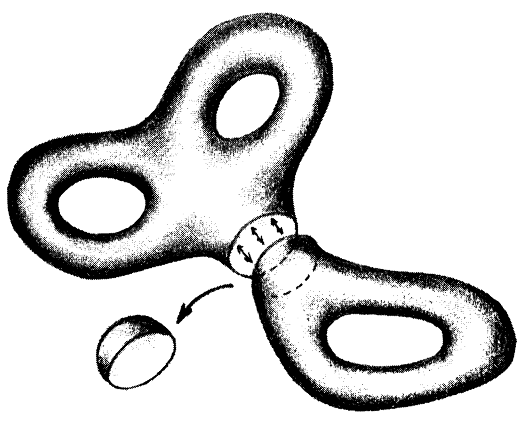


Mathematicians love to do that, it's one of the things they get a high on. [*Laughter.*] If you get a high by making pants and sewing them together, then by definition this is called doing topology, and you are a topologist.

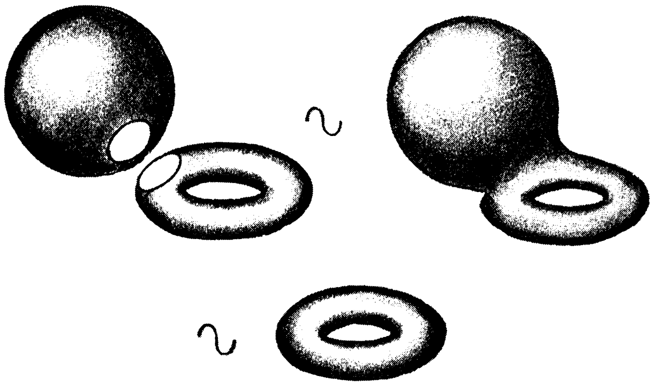
Even if you don't have a boundary, you can create one to define still another kind of sum. Up to now, we did only sewing, but you can also do some surgery. Take a surface, like this, nice and smooth, without boundary. Then I cut off a disc.



This yields a boundary, a circle which was not there before. I want to do the same thing with another surface, to create another circle. No I am in the previous situation, I have two surfaces with boundaries, and I can take their sum along these circles.



In this way I can define the sum of surfaces without boundaries. If I do this sum, between any surface and the sphere, then I find the same surface, up to equivalence. One can say that the sphere is the neutral element for this kind of sum.



On the other hand, if I have a surface with two holes, and a surface with one hole (so a torus), and I take their sum, then I obtain a surface with three holes. If I take the sum of a surface with three holes and a surface with one hole, then I get a surface with four holes, and so on.

One says that a surface is irreducible if, when you express it as a sum of two surfaces, one of them is necessarily a sphere. The torus is an irreducible surface, and every surface which is not equivalent to the sphere can be expressed as a sum of toruses, a certain number of times, corresponding to the number of holes in the surface.

I repeat that in everything I have said, I meant the surfaces to be orientable. And we have just done part of the theory of such surfaces, which have dimension two.

Now I want to go on to objects having dimension three.

A while back, we spoke of people living in two dimensions, say on a surface. They are very small. What they see around them is also small, and it looks like a plane. But they could ask themselves: if we were able to see very far out, what would space look like? What about us? We are very small beings, on something which is three-dimensional. Are we living on something which is the analogue of the three-dimensional sphere? What happens if we look far out in space, do we find a hole? One can also ask the question in dimension two, but for us dimension three is more relevant.

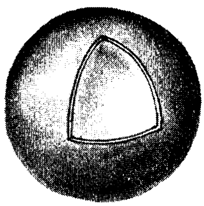
We see a three-dimensional space, and we have telescopes which are more and more powerful. If we can see sufficiently far, what are we going to find? Are we living on an object equivalent to a sphere? Or are we going to find holes? This is getting serious. You can really raise this question about the nature of the universe. So if you are dead set on wanting a physical interpretation for what I am doing today, there you are.

I started in dimension 2 because it was easier to define the notion of sum than in dimension 3.

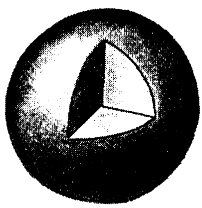
A GENTLEMAN. But the pants had dimension 3.

SERGE LANG. No, no! The surface of the pants has dimension 2. Of course, the pants exist in a three-dimensional space, but the surface itself has only two dimensions. You must distinguish between the dimension of the object itself, the surface itself, and the space in which it is embedded. Now it's the objects themselves which have dimension three.

Take the ball, for instance, the interior of the sphere, the full ball. It has dimension three.



the sphere



the ball

This ball, without the sphere which is its boundary, is equivalent to the whole three-dimensional space, for the same reason that the interior of the disc is equivalent to the plane R^2, or the interval without boundary is equivalent to the line R. The letter R denotes the real line, the straight line, and I put a small 2 on top to indicate that the space has dimension 2. For three-dimensional space, I would write R^3.

Something of dimension 1 is a curve. Something of dimension 2 is a surface. What do you call something of dimension 3?

AUDIENCE. A volume.

SERGE LANG. If you wish. But the word "volume" has several meanings. It can mean space itself, or it can mean the numerical value of this space. For instance, the interior of a suitcase, you might say that it is a volume, but you could also say that it is three cubic feet. You have to distinguish the two notions.

In rubber geometry, I don't measure a volume with numbers, because something can be equivalent to something else which is much larger, just by pulling and stretching.

I could go on talking about three-dimensional things, but they have a name in mathematics, a technical name. They are called manifolds, three-dimensional manifolds. I don't like this name, but that's the way they are called. Again I have the notion of rubber equivalence; I have the notion of compact manifold in other words, a manifold which does not go to infinity no matter how you deform it. I also have the notion of boundary, which will be what?

LADY. A surface.

SERGE LANG. Right, perfect. You have understood what I'm talking about.

OK, I've been talking for an hour. For the last two years, we stopped after an hour, then there were some questions, and people stayed around for quite a while. But I have to allow people to go if they want to. So we can have an intermission for a few minutes. The main topic which I want to discuss has to do with the classification of three-dimensional manifolds, and even of some non-compact objects.

In dimension two, I gave a theorem classifying surfaces: There is only the sphere and toruses with more and more holes. In dimension three, it's an extremely difficult problem, which mathematicians try to solve. This is precisely Thurston's contribution, to have stated a conjecture which describes all of them. There will also be sums, and there will be holes, but it will be much more complicated. That's what I want to do later.

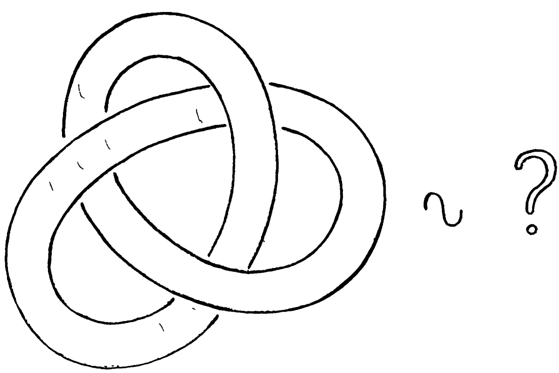
But for the moment, intermission or recess, depending on your rhetoric.

[*Applause. Someone asks if he has the time to go get a drink across the street, and I answer yes. We start again after about fifteen minutes.*]

The second hour

[*At the start, the room was full, with about 230 persons. About three fourths have now come back for this second part.*]

[*On the blackboard, one can see the following picture, drawn by someone in the audience.*]



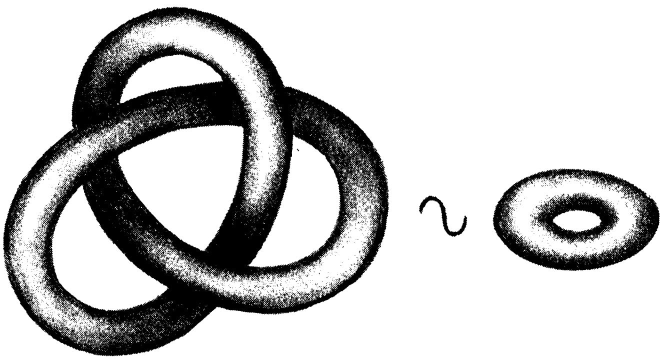
SERGE LANG. [*Looking at the picture.*] Ah, very good drawing. It's analogous to the knot, but with a torus. Do you have any questions on what I have done so far?

GENTLEMAN. Is this surface equivalent to a torus?

SERGE LANG. Very good question. What do you think?

SOMEONE. How many holes does this surface have?

SERGE LANG. Well, it's a surface with one hole, embedded in three-dimensional space, and one can't deform it on the torus if you ask that the deformation take place in three-dimensional space. But you can deform it on the torus if you allow higher dimensional spaces. It's just like the knot at the beginning. You see, there are several ways of representing the same surface.



GENTLEMAN. If someone walks inside the surface, then they have no way of knowing if the thing is knotted or not.

SERGE LANG. Yes, excellent remark. It's just like the knot. If you move forward on the knot, always walking ahead, then you come back to where you started from, but you don't have any way of knowing that you are not on the circle.

[*A hand goes up.*]

SERGE LANG. Yes?

A LADY. What about the Moebius strip?

SERGE LANG. I have already said that I want to consider only orientable objects, precisely to eliminate this kind of thing, because I wanted to avoid technicalities to make simpler statements. It was to protect myself against someone who would complain that I was being incomplete. If I discuss non-orientable surfaces, then I won't have time to talk about three-dimensional things, and that's what I want to talk about. OK, the Moebius strip, many of you probably have heard about it, and there isn't much point dealing with it now. But you probably don't know so much about three-dimensional objects.

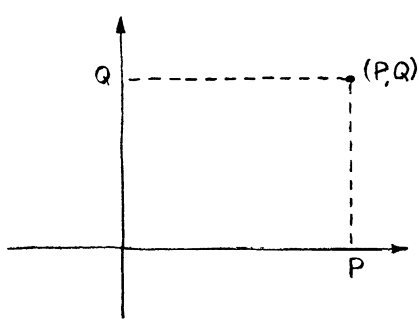
Besides, they are relevant for the world in which we live. I have already said that mathematicians work with lots of possibilities, lots of models. As mathematicians, we are interested in the beauty of these models, and not necessarily in their physical applications. Today, I classified surfaces, and I am interested in the classification of three-dimensional manifolds. I am trying to describe them all. After we know them all, then we can ask which ones correspond to the physical world, the world we live in. A physicist chooses among these models to find those which fit the empirical world. I myself have never done physics, and it disturbs me that there is a correlation between the world of experience, the world with which we come into contact with our senses. I have always felt that way, ever since I was a student. I have no ability in physics, which does not really interest me.

A LADY. No wonder students have a hard time to apply what they know in mathematics in order to do physics.

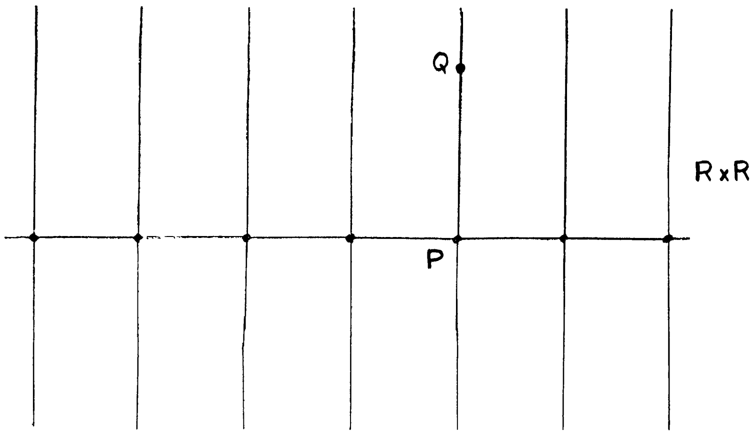
SERGE LANG. I have no reason to hide my personal tastes, but I don't impose them as being a law of nature. I like the classification of things, just like that, I make up models, and I tell the physicist: "Pick the one that suits you." On the other hand, there are other mathematicians, like Atiyah, or Singer, who are directly interested in physics. Conversely, there are physicists who understand mathematics very well, and who do both at the same time. And I am all for it. I make that quite clear to students, and I encourage them to do both if they are able to do so, and if they like it. But everyone has his own limitation.

OK, let's return to three dimensions. It becomes a lot harder to draw, because for instance, even the three-dimensional sphere, I can't show it to you. The ordinary sphere, the two-dimensional one, I could draw it as the set of all points which lie at a certain distance from a given point, which is called its center. The three-dimensional sphere S^3 again can be defined as the set of points in four-dimensional space, which are at a given distance from the center. So the sphere S^3 is embedded in four-dimensional space, and we can't draw it. But we can conceive it.

What I can do, however, are drawings which suggest what happens. Or give other representations. For example, in the plane, take two axes and points P, Q on these axes.



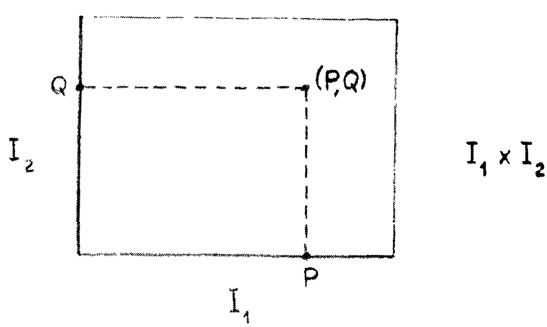
I can write the point (P, Q) where Q is on a line R and (P, Q) is in the plane. This construction is called a product. It is as if I put a line above each point P, and the point Q wanders along a line above P.



As I said, this construction is called taking a product. We saw that the plane R^2 is a product of R with R, and we write

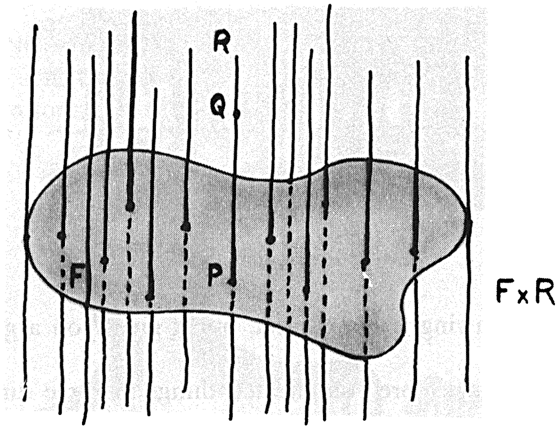
$$R^2 = R \times R.$$

Similarly, if I have two intervals I_1 and I_2, and if I take the set of all points P in the first and Q in the second, then all the pairs of points (P, Q) like this constitute all the points of a rectangle.

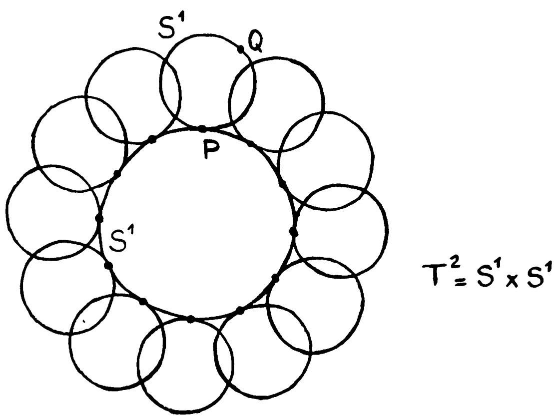


I can construct products like this with any two sets. If I take a surface F having dimension two, I can take its product with any one-dimensional thing.

If F has dimension 2, then the product is a manifold of dimension 3. It is the set of points (P, Q) where the first point P is a point on the surface F, and the second point Q belongs to a one-dimensional space.



Let me draw another example. Let S^1 be the circle. I take the product $S^1 \times S^1$, that is the set of all couples (P, Q) where P is on the circle, and Q is on another circle. To each point P of the first circle, I can associate all the points on the other circle.



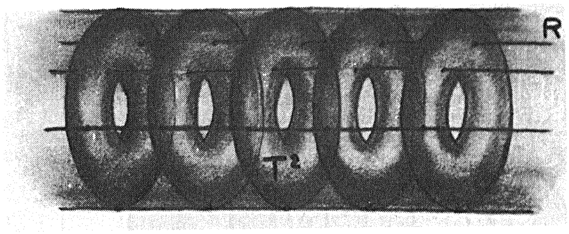
What kind of a surface do I get?

AUDIENCE. A torus.

SERGE LANG. Right, a torus. $S^1 \times S^1$ is a torus. I'll call it T^2, with T like torus, and the 2 because it has dimension two.

This notion of product allows me to construct higher dimensional object, and I can write them down. I don't need to draw them any more.

Now I can get to bigger dimensions. Take T^2 for instance, and its products with R, its product with a straight line. I would have a hard time drawing it, but I can represent it by drawing a torus, straight lines like that, and I can consider the torus as a section of this thing, this three-dimensional product.



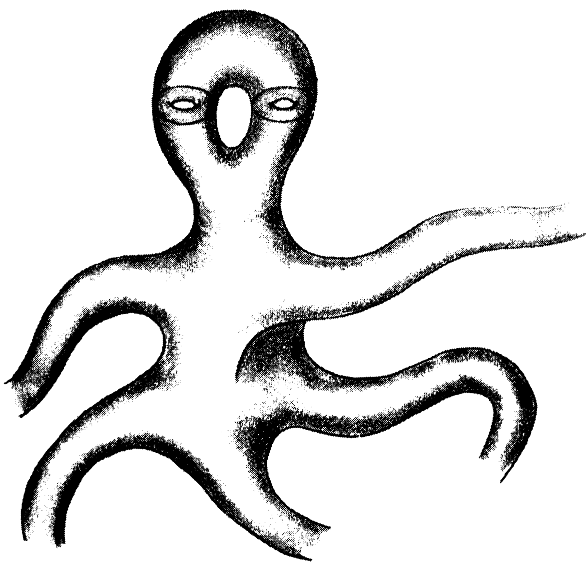
$T^2 \times R$

Of course, this drawing is not correct, but it gives you a good idea of what's happening.

Next I want to draw more complicated things in three dimensions. I already have the sphere in three dimensions, and the product $T^2 \times R$, but I want things which correspond to surfaces, when I had holes. What does such a thing look like?

LADY. It could be a pipe with thick walls.

SERGE LANG. That's right. I want holes, toruses, and things which go to infinity. [*Serge Lang draws the following picture.*]



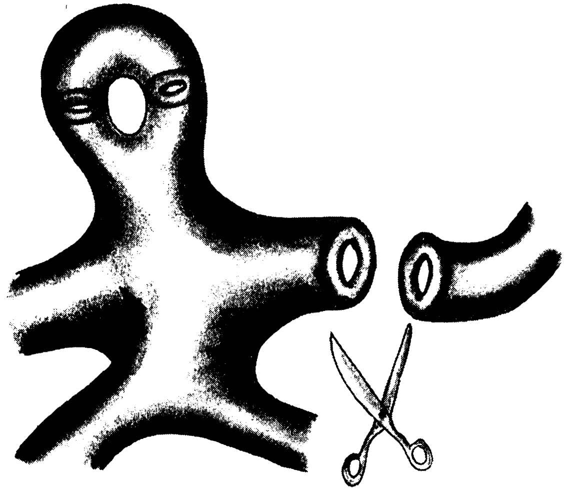
[*Laughter.*] So, here is a three-dimensional thing. What do I call it?

AUDIENCE. An octopus!

SERGE LANG. Precisely, an octopus. In dimension two, I had pants. In dimension three, I have octopusses. This suggests . . . suppose I take a pair of scissors, and I cut one of the legs. What do I get?

AUDIENCE. . . .

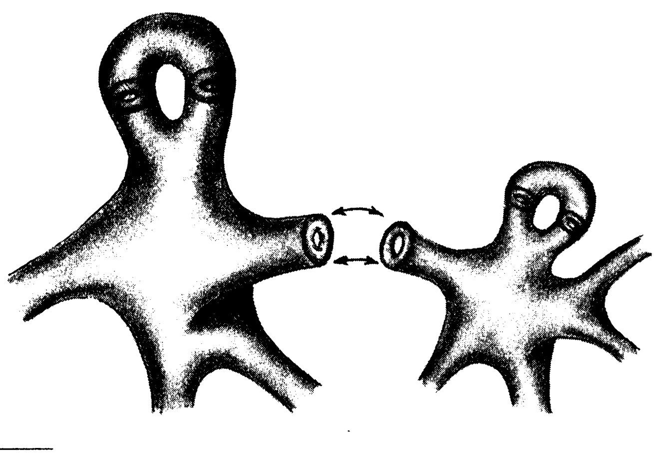
SERGE LANG. The octopus does not have a boundary.[3] If I cut one of its legs, I get something whose boundary will have dimension two, and which will be a torus.



Now suppose you have some three-dimensional thing, whose boundary is a torus, and you have another thing whose boundary is also a torus. You must have an irresistible impulse to do something to them. What is your irresistible impulse? You [*pointing to a lady*].

LADY. To glue them together.

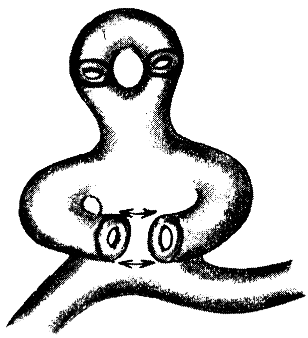
SERGE LANG. Right, just like before with circles. So I take two octopusses.



[3] Unfortunately, one cannot draw the picture correctly, and more than one could draw the sphere S^3. The way we have made the drawings, there is a boundary, but nevertheless, they show rather well what's going on.

I cut one leg of each one, I get two toruses, and I glue them together. Then I have taken the sum of the two octopusses along a torus.

I can also do a similar operation with only one octopus, by cutting two legs, and glueing the two sections, which are toruses.



Before we also had caps. What do we have now? I have a boundary which is a torus, and I want to eliminate the boundary. What do I do?

GENTLEMAN. You glue a ring.

SERGE LANG. Very good, that's right, a ring, the interior of a torus. I take the ring, and I glue it on the torus. It's the same type of operation as capping, but with one more dimension. So I have eliminated a piece of the boundary.

GENTLEMAN. How many legs can an octopus have?

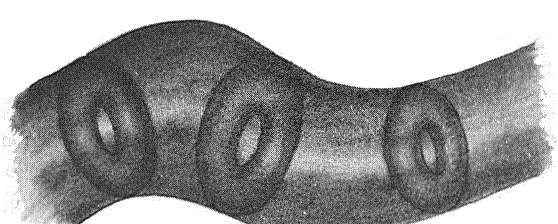
SERGE LANG. Any number. Two octopusses can have a different number of legs, in which case they are not equivalent. [*Laughter.*]

GENTLEMAN. How do you take the sum of two octopusses if one of them has an odd number of legs, and the other has an even number of legs?

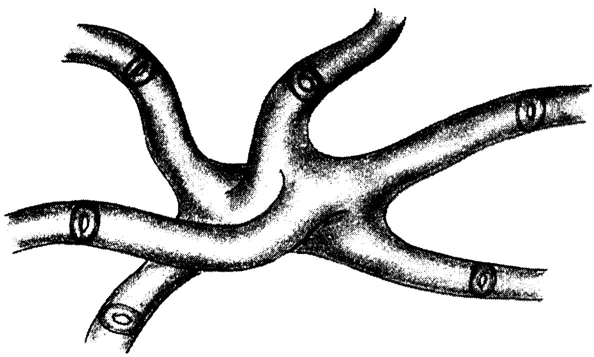
SERGE LANG. I didn't say that you had to glue all the legs of one to all the legs of the other. You can just glue together some of the legs, and then you can cap the rest of them with solid rings.

GENTLEMAN. What about $T^2 \times R$?

SERGE LANG. Well, $T^2 \times R$, it's . . . euh . . . it's like an octopus without holes, which has only two legs.



If I cut $T^2 \times R$ by making a section, then I get a boundary which is a torus. And there also exist octopusses without holes, with several legs.



Just like for surfaces, one says that an octopus is irreducible if the only way to express it as a sum of two octopusses is when one of the two is equivalent to $T^2 \times R$, or when it is a capping operation, so glueing a ring like the gentleman said a minute ago.

If I take an octopus and take its sum with $T^2 \times R$, then I get an octopus which is equivalent to the one I started with. From the point of view of equivalence, I have not changed anything. One can say that $T^2 \times R$ is the neutral element with respect to this sort of addition, obtained by cutting and glueing a leg.

Now it is perfectly conceivable that after a finite number of additions like this, I can eliminate all the legs. Let me write this down.

> After a finite number of additions, one can eliminate, in many ways, all the legs. Then one obtains a three dimensional manifold, compact, without boundary.

LADY. But there are holes.

SERGE LANG. Yes, definitely. We have eliminated the legs, but we have created holes, and there can be many of them. This is one of the ways of constructing three-dimensional manifolds, compact and without boundary . . . and orientable, just so my conscience does not bother me, and so nobody complains.

Of course, the next time you are on the beach, you can try it out [*Laughter*], take the legs of an octopus, and you can even knot them before you glue them together.

In order to classify octopusses, we must therefore classify the irreducible ones, and then we must classify the way you can add them together, as I did just now by cutting and glueing their legs.

Up to now, I have described geometric models: first models of surfaces, then models of three-dimensional manifolds, octopusses, the three-dimensional sphere S^3, which is not an octopus, and which is something else.

AUDIENCE. It does not have any holes.

SERGE LANG. Right, no holes. Then one can raise the following question.

Take all compact, three-dimensional manifolds, without holes and without boundary. Can you describe all of them? The problem is unsolved. Poincaré's conjecture is that a three-dimensional manifold, compact, without holes, without boundary is equivalent to the sphere S^3. Presumably there is no other. Of course, one should make more precise what is meant by a "hole", but let's leave this technicality aside for today.

Many people have tried to find the answer to Poincaré's conjecture, but so far no one has succeeded. In 1960, Smale proved the analogous conjecture in dimension bigger or equal to 5. After that, there remained the problem in dimension three and four. But the smaller the dimension, the harder it becomes, because you have not enough room to move around. The case of dimension 4 was just solved by Freedman in 1981. Many mathematicians contributed to the solution. They developed the theory as far as they could by "pure thought", without too many technical complications, and then they got stuck.

And it was Freedman, after six to ten years of work, who got it. It was very difficult, and very technical, and very complicated. It is one of the great result of contemporary mathematics, it is a first rate result.

There remains the three-dimensional case.

Therefore I cannot state a complete classification for three-dimensional manifolds, because Poincaré's conjecture is not yet proved.

For the other three-dimensional manifolds, there is a conjecture due to Thurston, of which he has proved a good part himself: it is possible to make up a concrete list, not too big, of certain manifolds, such that:

> Every three-dimensional manifold, without boundary, compact and orientable, is either in this list, or is a sum of octopusses.

So far, I have carried out the part I wanted to do concerning rubber geometry. To make Thurston's conjecture more precise, and to describe the list more precisely, I have to deal with entirely different ideas.

And it is rather interesting, it is even very interesting, that the manifolds in this list will be constructed by the same method which will also allow us to construct octopusses with legs. In other words, we shall construct simultaneously, by the same process, manifolds without legs and manifolds with legs. To do this, we must leave rubber geometry, and do an entirely different kind of geometry. Most of you probably have already heard of it, non-euclidean geometry. But we have to do it in dimension three.

Before I go any further, do you have any questions? How do you feel about all this?

A GENTLEMAN. Through any point of an octopus, is there only one torus or are there several toruses?

SERGE LANG. It depends. If I cut a leg to make a section, then I get a torus. But if I cut elsewhere, it depends. I have to cut in the right place to get a torus, I have to cut a leg. A mathematician would say it this way:

> An octopus is a three-dimensional manifold, non-compact, without boundary, with a finite number of ends, each of which is equivalent to $T^2 \times R$.

So if I cut near a point, and not a leg, if I cut elsewhere besides an end which is equivalent to $T^2 \times R$, then there is no reason why I should get a torus. In fact, if I cut near a point, I can cut a ball, just like when you take an ice cream ball, and it leaves a boundary which is an ordinary sphere. You can also think of an air bubble in a piece of swiss cheese. It's the same thing as for surfaces. In that case, I cut a disc, leaving a circle as boundary, and that's how I defined the sum of two surfaces by glueing together two circles.

By cutting off balls, I can add together three-dimensional manifolds. I cut off a ball in the first, I cut off a ball in the other, this leaves a boundary in each one; a sphere in each one. I glue the two spheres together, and I obtain the sum of the manifolds. One says that a manifold is irreducible if, when I express it as a sum like this, one of the two must be equivalent to a sphere S^3.[4] In 1962, Milnor proved that every compact, three-dimensional manifold without boundary can be expressed as a sum of irreducible manifolds, essentially in a unique way.[5] This result reduces the classification of three-dimensional manifolds to the classification of irreducible manifolds. Always up to equivalence, of course.

Are there any other questions? No? OK, then we go on, and we come to the geometry of distances, and non-euclidean geometry. But I have been talking for two and a half hours. What do I do with the non-euclidean stuff? Do you want to leave? Have you had enough? I'll do what you want.

A LADY. No, we stay, you have stimulated our curiosity. We go all the way.

SERGE LANG. Oh, I have stimulated your curiosity! Then the octopusses sank in. Good [*laughing*], do you want another five minute recess and we go back to work?

AUDIENCE. No, we are all set, let's go on.

---

[4] Note that we use the word "irreducible" here with respect to the sum taken along spheres, while we already used this word when dealing with the sum along toruses. There are indeed two types of sums, and the context should always make precise which one is meant.

[5] J. Milnor, "A unique factorization theorem for 3-manifolds," *Amer. J. Math.* **84** (1962).

GENTLEMAN. Now that we are coming along, keep it up. [*And the gentleman makes a gesture meaning "keep it up".*]

SERGE LANG. OK, then, let's go. But you have some stomach to do mathematics! If any one wants to go, or has an appointment, don't be afraid to leave. [*Laughter.*] It's not that I want to kick you out, but still . . .

[*Several persons leave, and others will continue to leave during this last hour.*]

The third hour

Now I leave rubber geometry to do the geometry of distances. On the real line, or in the plane, or in ordinary three-dimensional space, we have the notion of distance. We are then interested in a new type of equivalence, which preserves distances.

I shall call motion a transformation which preserves distances. We shall have motions in euclidean geometry, and also in non-euclidean geometry, but I want to start with examples in the euclidean case, just to give you the idea. Using these motions, we can then do certain identifications, which will allow us to recover octopusses, and the geometry of distances will thus meet the rubber geometry. So we are going to do something quite substantial.

Let's start with the straight line R, with 1, 2, 3,..., -1, -2, -3, ...

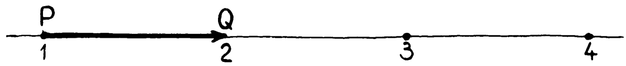
Suppose given a certain direction, and a certain distance which I denote by an arrow.

Then take a point P. I can move it in the direction of the arrow, exactly this distance. Then I get a point Q which I call the translation of P, and which I write $\tau(P)$.[6]

$$P \longrightarrow Q = \tau(P).$$

For concreteness, take the arrow to have length 1. Then the translate of 1 is 2; the translate of 2 is 3; and so on. Now I identify a point P with its translations. Let me draw a point and its translations.

[6] The letter τ is a greek letter, tau. I would use a T except for the fact that we have already used T for a torus, so we need another letter.



If I identify like this, then I obtain a circle, just in the way that you wanted to do it at the beginning. If I take an interval, with its end points, and I identify the end points, then I get a circle.

It was a very healthy reaction, a very mathematical reaction, for you to want to make these identifications, except that we are using them now. They were not allowed when we defined the notion of equivalence in rubber geometry. So, briefly, on the line, I get a circle by identifying a point with its translations in a given direction, by a given distance.

Of course, if P is a point, then I identify P with the next point $\tau(P)$, then with the next $R = \tau(Q)$. And how can I write R?

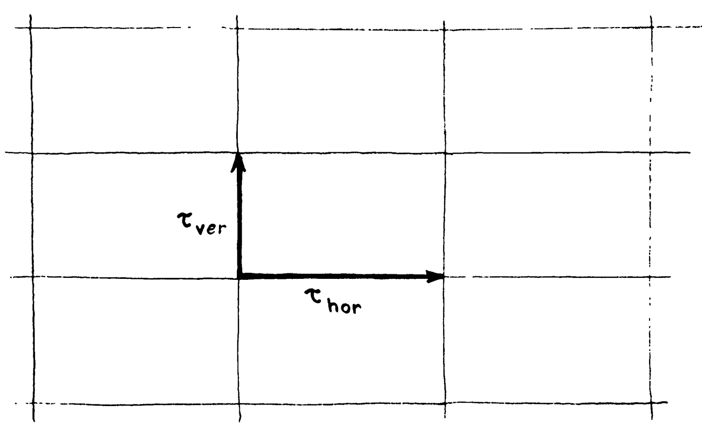
AUDIENCE. $\tau(\tau(P))$.

SERGE LANG. Right, $\tau(\tau(P))$, which I also write $\tau^2(P)$. And if I iterate a third time, then I write

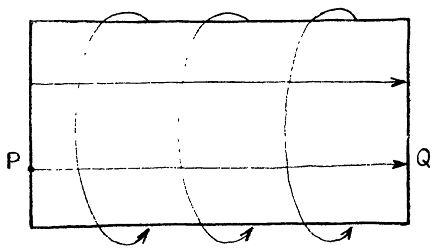
$$\tau(R) = \tau(\tau(\tau(P))) = \tau^3(P).$$

And if I go in the opposite direction, then I write $\tau^{-1}(P)$.

All right, let's go to dimension 2. Then I have vertical translations, and horizontal translations, which I denote by τ_{ver} and τ_{hor}.



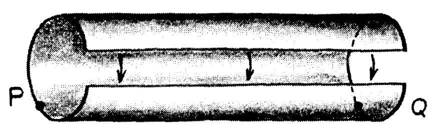
I can then make identifications, or translations, in two directions: the vertical direction, and the horizontal direction. Suppose that I identify the point P and the point Q in this next diagram. I identify the left hand side and the right hand side of the rectangle. And I also identify the top side and the bottom side.



What do I get?

LADY. A sphere.

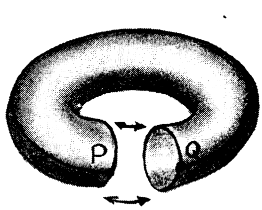
SERGE LANG. No! Watch out, to identify means what? When I identify the top and the bottom, then I get a cylinder.



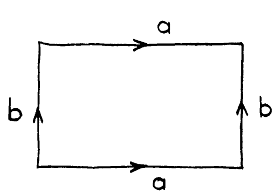
Then if I identify the sides, what do I get?

AUDIENCE. A torus.

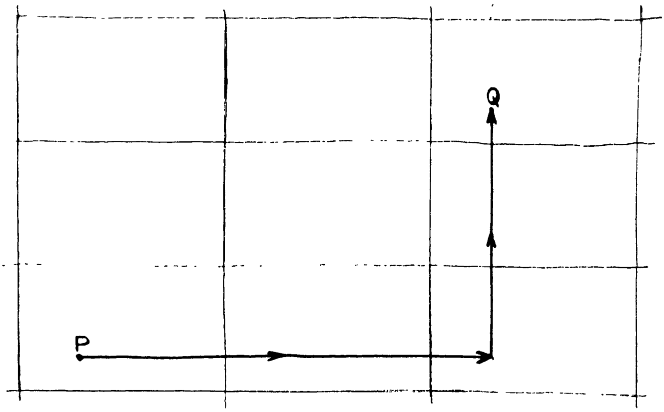
SERGE LANG. That's right, a torus, T^2.



Now you see that I can describe the torus by means of a diagram in dimension two, by identifications and translations horizontally as well as vertically.



I can also make these identifications in the whole plane.



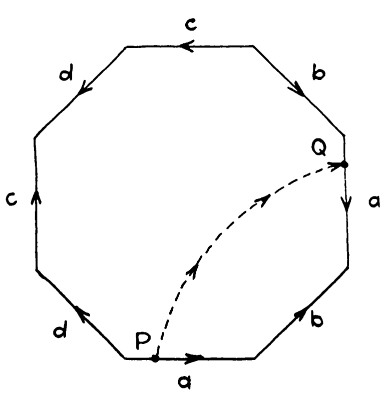
I'll say that two points of the plane are equivalent if I can make horizontal and vertical translations which move one point on the other. But this is a different kind of equivalence from the rubber equivalence. Here I have the notion of direction and distance in addition.

I now need these two notions, which I had completely disregarded previously. So I have to specify which equivalence I mean, and I need two different words to denote these two equivalences. I have to fix some terminology, which I am going to do more systematically in a moment.

OK, I just got a torus, that is a surface with a hole, by making certain identifications. If I want a surface with several holes, can you guess what I should identify and how? Here I got a torus with a rectangle. If I want a surface with, say, two holes, what kind of identifications should I make?

LADY. Draw another line in the middle, or something like this.

SERGE LANG. Yes, you are right, one should draw more lines, but not quite as you said. Let me show you just what to do. Instead of four sides, use a polygon with eight sides.



And make the identifications just as I drew them. For instance, I drew the point P identified with the point Q.

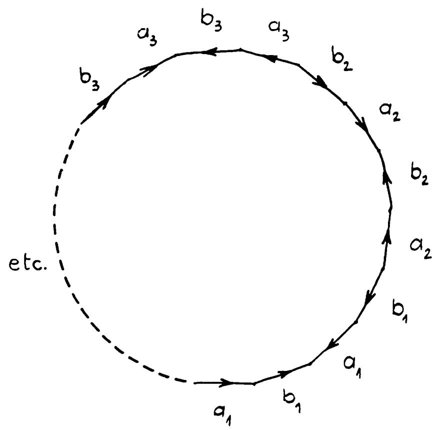
And if I want a surface with three holes?

AUDIENCE. Use a polygon with twelve sides.

SERGE LANG. Right, eight for the surface with two holes, and twelve for the surface with three holes. And if I want a surface with n holes, then I need . . .

AUDIENCE. $4n$.

SERGE LANG. That's right, and we draw it like this.



And that's how we get a representation of a surface with several holes.

GENTLEMAN. And if you have a polygon with six sides?

SERGE LANG. It won't give necessarily a surface with holes like the ones we had before. It can give something else, a non-orientable surface, but today, I want to limit myself to orientable surfaces.[7] But your question shows that you have understood what I am talking about.

You see, the torus T^2 can be obtained as a quotient of the plane, by means of certain identifications, which I am going to write with a slanted bar, on the left hand side:

$$T^2 \sim \text{Identif.} \setminus R^2.$$

These identifications were translations.

GENTLEMAN. What does the 2 mean?

[7] It all depends on the respective position of the sides which are to be identified, and of their orientation. In some cases one can find a torus, and in other cases, one can find a non-orientable surface. This is a good exercise: study those surfaces obtained by identifying sides in a polygon with $2n$ sides.

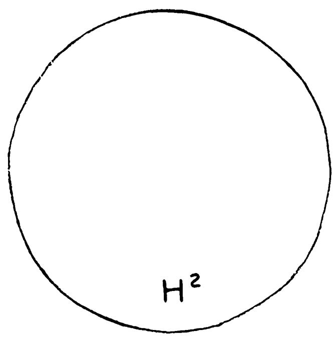
SERGE LANG. It's just to denote the dimension. The numbers which I write as superscripts, upstairs, are just to indicate dimension. I did not use numbers in any other way. I had sworn not to use numbers, but it is still useful to write the little 2 upstairs. I said I would do only geometric things, but the 2 denotes dimension. Do you allow this?

AUDIENCE. Yes.

SERGE LANG. Thanks. It's because I had promised not to use numbers. But this 2 is not really a number. [*Laughter.*]

Good, so I have represented T^2 as a quotient of R^2 by translations. And this was euclidean, with respect to translations. Now let's go to non-euclidean motions.

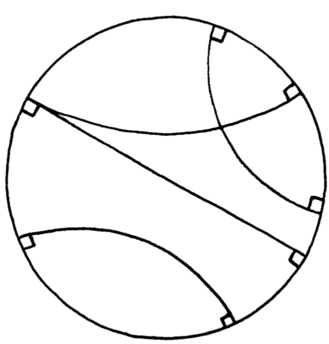
One of the models of the non-euclidean plane is the disc. I'll call it H^2, H for hyperbolic.



We need the notion of hyperbolic distance, and the notion of "line" with respect to this distance. In an audience like today's, there must be some of you who already know about this. Who knows already what this means?

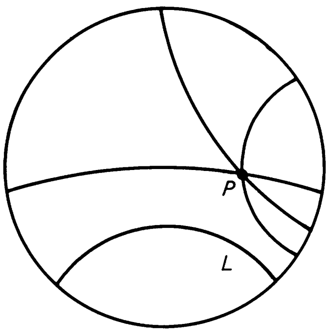
AUDIENCE. ???

SERGE LANG. All right, I'll tell you what it means. By definition, a hyperbolic line is just an arc of circle in H^2, perpendicular to the boundary. I can draw it like this. Here are some hyperbolic lines.

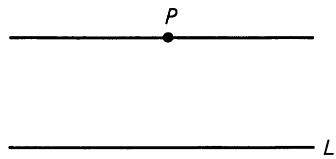


Some of them intersect each other, and others do not. Perpendicularity means the same thing as in the euclidean case.

As you see, you can have infinitely many lines passing through a given point P, but not intersecting another given line L. This cannot happen in the euclidean case.



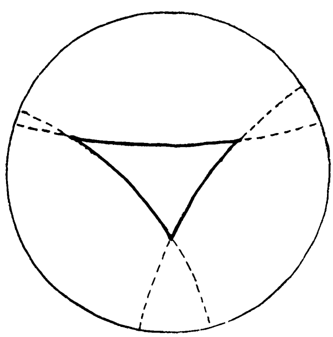
non-euclidean case



euclidean case

In euclidean geometry, given a line L and a point P, there is just one line passing through P and parallel to L.

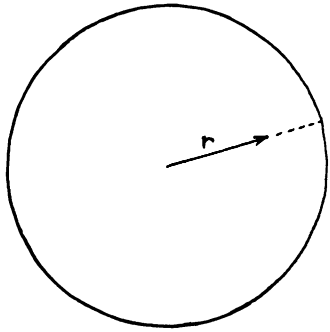
We define a triangle just as in the euclidean case. Here is an example of a triangle, whose sides are line segments.



Now that we see what lines look like, we are going to describe the notion of hyperbolic distance, and the spaces we get from this point of view. After that, we'll make the connection with octopusses, and the classification of three-dimensional manifolds. I want to end up by stating Thurston's conjecture.

So we have to define a new distance, called hyperbolic distance. It's also called the Poincaré distance by the French, and the Lobatchevski distance by the Russians. I call it hyperbolic distance, so nobody gets upset. [*Laughter.*]

To describe this hyperbolic distance completely, I would need some formulas, and I don't want to write down formulas which are too technical. But I can speak of the rate of change of the distance when I start from the center, and move toward the boundary. This means that if r is the euclidean distance from the center,



then the rate of change of the hyperbolic distance along a ray can be given by a very simple expression, which is

$$\frac{1}{1 - r^2}.$$

Here I supposed that the radius of the circle is equal to 1. So if I start from the center of the disc and I go toward the boundary along a ray, then how does the rate of change behave? You see, if r approaches 1, then r^2 also approaches 1, and $1 - r^2$ is very small. Then the fraction is very large. Therefore the rate of change of the distance becomes very large when I come near the boundary, and hence the distance becomes bigger and bigger. There is a formula which gives the distance in terms of r, with the logarithm. Who has heard of the logarithm?

[*Several hands go up.*]

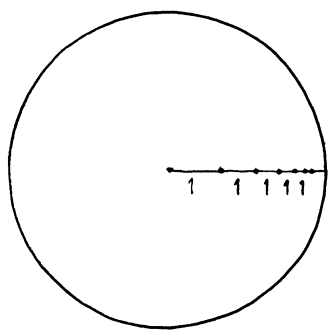
Ah! Several of you do know. Then I'll give the formula. Those who don't know what the logarithm is don't need to listen. The hyperbolic distance along a ray is

$$d = \frac{1}{2} \log \frac{1 + r}{1 - r}.$$

So the distance becomes larger and larger as we get nearer to the boundary.

You see that this is analogous to Einstein's thing, and to the way the world is made up. Suppose we start at the center, and we start moving toward the boundary, as far as possible. What happens when we go very far out, in our own universe? We know that the euclidean model fails, we know that space is going to get curved, a little like the hyperbolic lines a while back. And we speed up. Suppose that a ray of light goes in the same direction. If I measure its speed, then I find 186,000 miles per second. Now suppose I go faster. If the model was euclidean, then when I measure again, I should find a smaller value for the speed of light. Right? Well, the answer is no, not at all! I always find the same value. If two trains were going at the same speed, in the same direction, then they would not move with respect to each other. But this does not work the same way with light. Light always travels at the same speed. And the reason is that as I go faster and faster, then I grow smaller and smaller, and my measuring apparatus will also grow smaller, and when I measure the speed of light, then I find a constant value.

In the hyperbolic plane, we meet an analogous phenomenon. Here I have drawn points at a distance of 1 unit between each other, along a ray.



So how do you know that we don't live inside something like this? The further we go, the less we can know what happens on the other side–or even if it means anything to speak of "another side".

But we can still raise the question: what kind of universe do we live in? Then the mathematician creates models, and the physicists figure out which of those models fit the world we live in. It is not clear what we mean by "other side". By conceiving the hyperbolic plane in another way, not embedded in the ordinary plane, but intrinsically, by itself, there is no "other side". One of the possible questions is whether our universe is embedded in another one. But then we could not have any direct contact with this other universe, and we would have to deduce its properties only by its effect on our own universe.

All right, let's go back to mathematics. I have this model, and I can make identifications, just like in the euclidean model.

[*A hand goes up.*]

SERGE LANG. Yes?

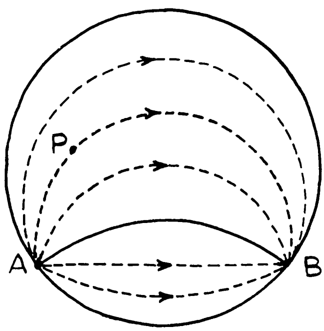
GENTLEMAN. You defined the distance with respect to the center, but can one define the distance also for any two points?

SERGE LANG. Yes, of course, but it would be much more technical to do so, and the formula would be more complicated, so I don't want to do it. I would need hyperbolic functions to write it down.

OK, so I want to make identifications. I need certain types of motions, which preserve the hyperbolic distance. As before, I can define translations. Suppose I am given a hyperbolic line. It gives me a direction, and I have the notion of hyperbolic distance. Take any point P. Where are its translations in the direction of this line? Along which curve does the point P move, in the direction of the line?

AUDIENCE. ???

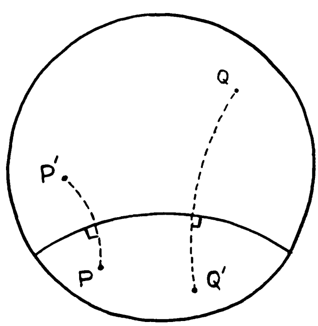
SERGE LANG. Well, let A and B be the two end points of the hyperbolic line, as shown here on the figure. The translates of P in the direction of the line are going to be on the arc APB.



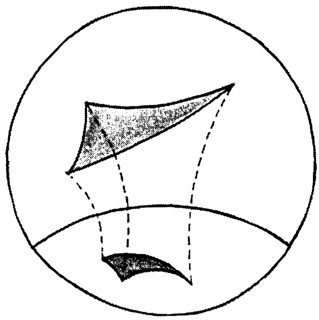
I can translate in one direction, or in the reverse direction, or iterate translations, and so on. Translations are examples of motions which preserve distance. There are others. Do you known which?

A HIGH SCHOOL STUDENT. Rotations, reflections.

SERGE LANG. Exactly. And in the hyperbolic plane, rotations are the same as in the Euclidean plane. As for reflections I am going to draw a point P and its reflection with respect to a line.



I can also draw a triangle, and its reflection with respect to the same line.



The reflection of a point with respect to the center is the same as in the euclidean case.

LADY. Then the hyperbolic plane has a center?

SERGE LANG. No, the hyperbolic plane by itself does not have a center, but the model which I gave for it has one. In the hyperbolic plane, the situation looks the same everywhere, no matter how close you are to the boundary for the euclidean distance. Given any two points P and Q there always exists a translation which moves P and Q. One says that the hyperbolic plane is homogeneous. In the hyperbolic plane, you are always infinitely far from the boundary. When I represent it by a disc, I choose a center, just as when I choose an origin for the euclidean plane.

OK, let's go back to identifications. I have rotations, translations, and reflections, and I can also combine them with each other, iterate them. In general, these give me all distance-preserving transformations. As I already said, I shall call them motions for short.

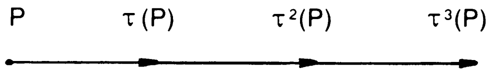
Now I still have to define another notion, that of a group of motions. I shall say that:

Γ is a group of motions if:

1) when two motions M_1 and M_2 are in Γ, then their composite $M_1 M_2$ is also in Γ.
2) the inverse of a motion M in Γ is also in Γ.

The composite $M_1 M_2$ is the motion such that, when you apply it to a point P , you get $M_1(M_2(P))$. The inverse of a motion which sends P to Q is the motion which sends Q to P. So now we have the notion of a group of motions.

I also need the notion of a discrete group. Let's start with an example, in the ordinary plane, and with translation. Take a point P, and translate it.



A STUDENT. The points are all distinct.

SERGE LANG. Yes, and what happens if I take a bounded domain in the plane?

THE STUDENT. Eventually, the points get out of the domain.

SERGE LANG. That's right. And I can do the same thing with a group. I will say that two points P and Q are equivalent with respect to the group Γ if there exists a motion M in Γ such that

$$Q = M(P).$$

This is an equivalence. And I say that Γ is discrete if, given a point P, among all the possible motions $M(P)$ with M in Γ, there is only a finite number lying in a bounded part of space. Essentially this means that in any bounded part of space, in any bounded domain, there is only a finite number of points which are equivalent to P with respect to Γ.

To distinguish this new equivalence from rubber equivalence, I have to mention Γ explicitly, so I could all it a Γ-equivalence for short.

Now suppose that Γ is discrete. I can identify points like that, with respect to Γ. I can identify all the points which are Γ-equivalent to each other. Then I obtain a new space after having made these identifications. And I denote this space by

$$\Gamma \setminus H^2.$$

This space will again be two-dimensional.

I have just done identification in dimension 2. Of course, I can also do identifications like this in dimension 3. What do we take as a model for hyperbolic space in dimension 3?

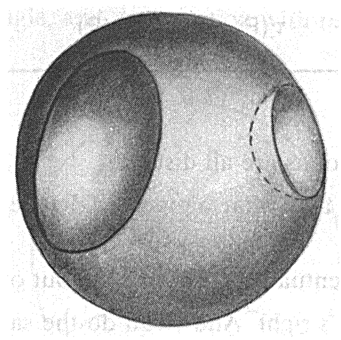
GENTLEMAN. The sphere.

SERGE LANG. Yes, the interior of the sphere, the ball. In dimension 3, we have H^3, which is the ordinary ball, but with a hyperbolic distance which is analogous to the hyperbolic distance in the plane. When you move toward the boundary, the distance becomes arbitrarily large.

And in hyperbolic 3-space, what do planes look like?

GENTLEMAN. They are parts of spheres?

SERGE LANG. That's right.



And we can also define translations, reflections, etc.

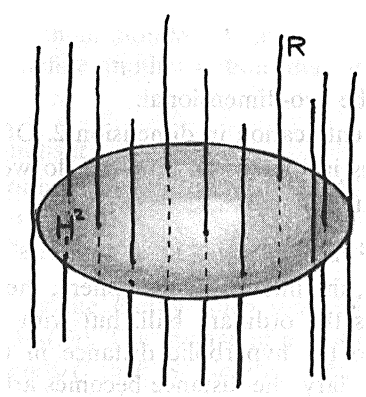
But there is also another way of constructing three-dimensional spaces, by using something having dimension 1 and something else having dimension 2. I have already used the product construction before. Now who can give me another example of a three-dimensional space which we could use by taking a product?

THE STUDENT. Take a line and a hyperbolic plane.

SERGE LANG. Ah! Very good! That's precisely what I wanted to get out of your head. So we have another example, by taking the product of the hyperbolic plane H^2 and the line R, which we write as

$$H^2 \times R.$$

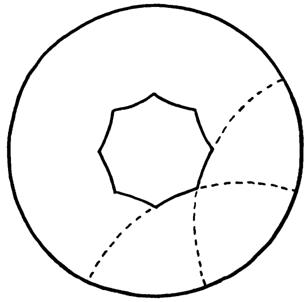
This space has the hyperbolic distance on H^2 and the ordinary distance on R.



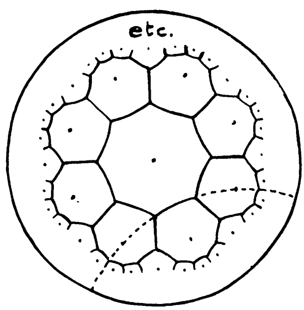
Now we have the fundamental examples

$$H^3 \quad \text{and} \quad H^2 \times R.$$

And we are ready to make the connection with rubber geometry, and the rubber equivalence. I am first going to recall a classical theorem on surfaces. Let's go back to our polygon, which we take in the hyperbolic plane. Its sides are hyperbolic line segments, but the polygon is equivalent to a polygon which we already drew, to construct surfaces by identifying certain sides of the polygon.



Then the hyperbolic plane can be covered by translations of this polygon, such that two translations are either disjoint, that is, they have no points in common, or they meet only along some side.



It's the same thing as when you cover the euclidean plane with squares, or rectangles, except that you cannot get a tiling of the euclidean plane by means of regular octagons, but you can tile the hyperbolic plane with any regular polygon.

Identifying certain sides as we did before amounts to making identifications with respect to a group of translations. And one has the theorem:

Theorem. Let F be a surface, compact, orientable, without boundary, and not equivalent to the sphere or to the torus. Then there exists a discrete group Γ such that the surface F is equivalent to the hyperbolic plane on which we have identified points with respect to Γ. In other words,

$$F \sim \Gamma \backslash H^2.$$

Well, this theorem dates back to the 19th century. And nobody, until Thurston, thought that there could be anything like it in dimension 3. It was Thurston's great discovery that there should be an analogous result, to conjecture it, and to prove it in certain cases. First I am going to state a result which connects the first part of the talk on rubber geometry, with the second part on non-euclidean geometry.

We always denote by Γ a discrete group of motions, but we suppose in addition that:

–For any point P, the only motion M in Γ such that $M(P) = P$ must be the identity, that is the motion which does not move any point.
–And to keep me honest, that the motions in Γ preserve orientation.

We always suppose that Γ satisfies these two extra conditions, even if I don't say so explicitly.[8]

Let Γ be a group of motions of H^3, for instance. Then we can have two cases.

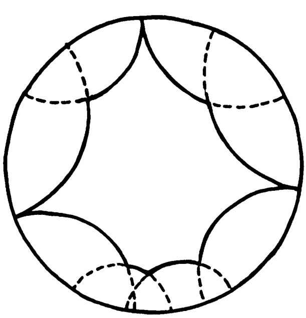
First case. $\Gamma \backslash H^3$ is compact.

In this case, the space we get by making identifications with respect to Γ is a manifold of dimension 3, compact, and without boundary. This is one of the ways of obtaining such manifolds.

Second case. $\Gamma \backslash H^3$ is not compact.

In this case, we have to use not only the notion of distance, but also the notion of volume which comes along with it. After I identify with respect to Γ, it is possible that the space $\Gamma \backslash H^3$ has finite volume. I will always suppose that Γ denotes a group such that the volume of $\Gamma \backslash H^3$ is finite.

Of course, you can have the same phenomenon in dimension 2. You can have a polygon whose sides go toward the boundary, and so the polygon has ends which are arbitrarily far from the center, as on the following figure.



[8] This is indeed the case for translations, and these conditions eliminate the possibility of reflections being in Γ.

But you can have a group of motions Γ, and even a group of translations, such that the surface which you obtain by making identifications with respect to Γ has ends which tend to infinity.

The same thing can happen in dimension 3 but it's too hard to draw. The parts which tend to infinity are sort of tubes.

Instead of H^3, I could also take $H^2 \times R$, and consider groups Γ such that $\Gamma \backslash (H^2 \times R)$ is compact, or is non-compact but has finite volume, with tubes going to infinity. What do these tubes look like?

Theorem. Let Γ be as above, a discrete group of motions of H^3 or of $H^2 \times R$, and let's make the identifications with respect to Γ. Suppose that the space you get has finite volume. Then this space is either compact, or is an octopus. Furthermore, $\Gamma \backslash H^3$ is an irreducible octopus.

In this theorem, it is understood that my group Γ satisfies the extra hypotheses I have stated above, for instance the volume of $\Gamma \backslash H^3$ or $\Gamma \backslash (H^2 \times R)$ is finite.

A LADY. In addition to the fact that Γ is discrete?

SERGE LANG. Yes, in addition, it's an extra hypothesis I have to make. After the identifications, I have to suppose that the volume is finite.

There is also a converse, which already gives some idea about the classification of octopusses.

Theorem. Every irreducible octopus is equivalent to a space of type

$$\Gamma \backslash H^3 \quad \text{or} \quad \Gamma \backslash (H^2 \times R).$$

And so I get back the octopusses! It's quite extraordinary. We started from an entirely different point of view, we made identifications in a geometry with distances, we took motions preserving the distance, and what do we find? Compact manifolds and octopusses! This is the connection between the first part with rubber geometry, and the second part with geometry of distances.

We are now coming to Thurston's conjecture. We have just seen two examples, H^3 and $H^2 \times R$, which are spaces with distances. I talked about a well-defined and short list of spaces. It consists of:

$$R^3, \quad S^3, \quad S^2 \times R, \quad H^3, \quad H^2 \times R,$$

that's five, and three others which I don't write down because it would be too technical to do so. There are eight of them altogether. Let's denote by X any one of these spaces.

Then Thurston's conjecture can be stated like this.

Conjecture. Let V be a three-dimensional manifold, compact, without boundary, and always orientable so we don't make things too complicated. Suppose that V is irreducible for the sum along spheres. Then V is equivalent to one of the following cases.

–There exists a unique X among the eight, and a group Γ such that X is compact and $V \sim \Gamma \backslash X$.
–V is a finite sum of octopusses, and each octopus is equivalent to some $\Gamma \backslash X$, where $X = H^3$ or $X = H^2 \times R$.

Besides, in the second case, there is a sort of uniqueness. More or less, this means that if we write V as a minimal sum of octopusses, then the expressions $\Gamma \backslash X$ which occur in this sum are essentially uniquely determined, up to an appropriate equivalence. It would be too technical to make this precise, and to define exactly what we mean by "essentially". One would have to define new equivalences, and this is not the time to do it.

So you see, to get octopusses, all you need is H^3 or $H^2 \times R$. This is the theorem which Thurston's is trying to prove, and which he has proved to a large extent.[9]

GENTLEMAN. What about Poincaré's conjecture?

SERGE LANG. S^3 is in the list, and Γ in this case is the identity. Poincaré's conjecture is isolated at one end of the list, and there is nothing you can do about it.

LADY. I have lost sight of something. What is the difference between R^3 and S^3?

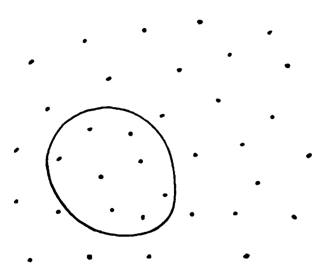
SERGE LANG. R^3 is not compact, it's the ordinary euclidean space around us. But S^3 is like the sphere, it is compact, while R^3 goes to infinity.

[*A hand goes up.*]

SERGE LANG. Yes.

LADY. Can you recall the definition of "discrete"?

SERGE LANG. "Discrete" means that if P is any point and we move P with all possible motions in Γ, that is you look at all points $M(P)$ with M in Γ, then in any bounded part of space there is only a finite number of such points.



[9] W.P. Thurston, "Three dimensional manifolds, Kleinian groups, and hyperbolic geometry," *Bull. Amer. Math. Soc.* Vol. **6**, 3, (1982).

That's what it means, that Γ is discrete. Plus the additional hypotheses I made.

A HIGH SCHOOL STUDENT. And if you take a group which is not discrete, what do you get?

SERGE LANG. Something disgusting, it's lousy. [*Laughter.*] No, it's more complicated, it's not a surface. If the group is not discrete, first you must ask that it be closed to get something which is half way decent. But if it's closed, then the dimension is going to go down. However, if the group is discrete, then there is lots of space between any two points which you identify, and the dimension stays the same when you identify points. If the group is not discrete, then it can give you something horrible–well, not necessarily horrible, but the dimension goes down. OK, you can look up what happens in books, to find out what goes on.

GENTLEMAN. A while back, you defined manifolds by using a hyperbolic geometry, that is the notion of hyperbolic distance. Was this just to fix ideas, or does Thurston's theory depend on the notion of neighborhood, open and closed sets, which are more general notions that that of hyperbolic distance?

SERGE LANG. When you speak of neighborhoods, you are dealing precisely with a type of geometry for which there are no distances. I made a list of eight geometries:

R^3, S^3, and $S^2 \times R$ with the ordinary distance;
$H^2 \times R$ with the hyperbolic distance on H^2 and the ordinary distance on R;
H^3 with the hyperbolic distance;

and three other cases, which have more complicated distances, which can't be called hyperbolic.[10]

---

[10] For mathematicians, I include here the description of these three geometries. One of them is $\widetilde{PSL_2(R)}$, where the ~ denotes the universal covering space. The last two are groups of matrices. One of them consists of the matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \text{ which is the Heisenberg group.}$$

The other consists of the matrices

$$\begin{pmatrix} a & 1 & b \\ 0 & a^{-1} & c \\ 0 & 0 & 1 \end{pmatrix}, \text{ with } a \text{ real and positive.}$$

The underlying spaces of these two are equivalent to R^3, but the distances are different from the usual euclidean distance.

THE GENTLEMAN. Then the topological properties of octopusses are closely tied up with these distances, whereas a priori they should be independent of them?

SERGE LANG. This is an excellent remark. Thurston's discovery was precisely that there was a connection between the two types of notions. This is what gives a lot of kick to his theory.

THE HIGH SCHOOL STUDENT. Yes, but you should then classify the discrete groups of the hyperbolic plane, otherwise, it won't work.

SERGE LANG. [*Laughing and very pleased*] He is absolutely right. What's your name?

THE STUDENT. Paul.

SERGE LANG. Indeed, what have we done here? We reduced something we didn't know to something else which we know—or which we don't know.

So, in the history of mathematics, it turns out that in certain cases, we know discrete groups pretty well, but not in other cases. One knows a lot of things about them, but many remain quite mysterious. But many people have worked on them, in the nineteenth century and the twentieth century. During the last thirty years, there has been a lot of progress about these groups. We know some of them very well. Similarly, one knows some three-dimensional manifolds quite well, and others not at all. So my answer is that, by reducing the study of manifolds to quotients of geometries with distances by means of discrete groups, one had the impression of making a huge step forward. My answer is therefore relative.

You see, in mathematics, it can happen that there are two things we don't know anything about, and we prove that one is equivalent to the other. This does not mean no progress has been made. The problems have been cut by half. [*Laughter.*] But this is not quite what happened here. One knew about three-dimensional manifolds in a certain way. One knew about discrete groups another way. In some sense, these ways were complementary. By putting them together, Thurston contributed to understanding both of them.

This does not mean that I personally know the classification of discrete groups. It's not my side of mathematics. I could learn it, but I do something else. I know some examples, and I could give you several if you want, but I don't know them well for the most part. There is no point getting a hang-up about not knowing them. There are lots of things in mathematics. When one needs something, one can always ask a friend to explain it, just the way I asked Walter Neumann to explain Thurston's theory to me.

GENTLEMAN. Let's go back to one less dimension. By identifying a square, you get a torus. But what about the sphere?

SERGE LANG. S^2 occurs on the side, you can't get it by identifying something, at any rate not the way I have described it here. Similarly, S^3 is on the side. That's Poincaré's conjecture, to prove that a three-dimensional manifold, which is compact, without boundary, and without holes, is equivalent to S^3. This conjecture remains all alone on one side of the theory. The difficulties which come up in connection with S^3 are different from those in connection with octopusses. Poincaré's conjecture is irreducible.

GENTLEMAN. And in lower dimension, for S^2?

SERGE LANG. For S^2, no problem, one knows the answer since the 19th century, that a surface of dimension 2, without holes, without boundary, and orientable, is equivalent to S^2.

GENTLEMAN. And one can get it from a representation of the plane . . .

SERGE LANG. No . . . euh . . . what kind of representation?

GENTLEMAN. With the disc H^2.

SERGE LANG. And with discrete groups? No. There is a theorem which says no. If you take the disc, with the Poincaré–Lobatchevski geometry, and you take a discrete group of motions, and you make identifications, then you will never find something equivalent to S^2. This is a theorem. Are you a mathematician?

GENTLEMAN. No.

SERGE LANG. Anyway, it's clear. A mathematician would have known the answer. [*Laughter.*] Oh no, no! No kidding, the question was very relevant, it is quite remarkable how well you react.

LADY. But Poincaré described two such geometries, it seems to me.

SERGE LANG. Well, we are coming back to the gentleman's question a while ago. He said one could put many distances on the same space. There is not only the distance I mentioned, in the hyperbolic plane, when the rate of change of the distance is $1/(1 - r^2)$. There are many other ways of defining distances. There is an infinite number of such ways. The study of such distances is called differential geometry. It consists in studying all possible ways of defining distances, and of introducing certain equivalences and classifying the distances up to such equivalences. But to do this would require an entire course in differential geometry. You are right, the subject is wide open in many directions.

LADY. But concretely, there is no realization . . .

SERGE LANG. Ah, concretely. But what one person finds concrete, another person will find abstract. It's entirely relative to your own brain, to what you know, to your talent in mathematics, to your intelligence, to your tastes, to your feelings. It's entirely relative. There is no absolute notion of what is concrete and what is abstract. For instance, what you might have found too abstract yesterday or today, could become concrete for you tomorrow.

If I draw enough octopusses, they will appear very concrete to you, it's a question of habit, partly. It depends on circumstances. There is no absolute answer. Of course, a mathematician could do something which others don't understand. The others could have the psychological reaction to find that it's too abstract, and they would say that instead of saying: "I don't understand."

LADY. It has no reality.

SERGE LANG. "Reality" where?

LADY. Physical.

SERGE LANG. Oh! The world of physics is much more extensive than you think. First, you know that if you take the three spatial dimensions, plus the time dimension, you already get four dimensions. And if you go very far, what do you find? Do you find octopusses? You find four-dimensional things? It could already have physical reality. Where do you stop with your physical reality? In what kind of space do we live? Is it curved? Is it an octopus? Is it something like H^3, or the ball with another metric? It's the physicist's business to find which space, and what kind of metric. It's for the physicist to choose between different models, which have been discovered by mathematicians, or to construct new ones which might fit better. Usually, people think that our space is homogeneous. Maybe this is not the case.

Take a point which wanders in space. Besides its spatial coordinates, there is a time coordinate, but there is also the speed, acceleration, curvature, which give me other parameters, other numbers, other dimensions. Take an electron which moves in space. At the same time, it turns, it wiggles, that gives me other dimensions. It's complicated to give a model for the electron, or even to know if the notion of electron makes sense. To describe those things that wiggle, elementary particles, you need other models, which may come precisely from differential geometry, among other things. Physics doesn't stop in any particular place! It's not just the physics of the drawings I can do here on the blackboard. And for other physical phenomena, maybe I need other models, which will appear too abstract for you today.

LADY. Yes, of course [*and the lady makes a gesture which shows that she has understood that those mathematical models which can be used in physics may come from any theory, no matter how abstract or advanced it may be*].

SERGE LANG. So, a good physicist is somebody who won't be scared by complicated models, who won't be chicken, who will seek his models in what engineers find too abstract. Except that the physicist will find a good model, and he will win. He will make it in the history of science precisely because he will liberate himself from the intellectual constraints of his colleagues, and will make concrete what others found too abstract. In other words, there are no limits. The only limits are for each individual, those of his own brain, his own temperament, his own tastes . . .

[*Serge Lang stops here and catches his breath.*]

Ouf! [*Laughing*] Some marathon!

[*Warm applause. After three and a half hours, there remain about 100 persons in the room.*]

Well, so this is goodbye. It doesn't happen every day, it's unique, to have been able to stay here like this for three and a half hours, with an audience like you. It's unique. I really appreciate it a lot. I was really pleased.