The conference

So, the conference, I think I'm going to talk about things in general for ten minutes, and after that, we'll try to do mathematics together. The talk will be as the title says: "What does a pure mathematician do and why?"

It's very difficult to explain "why" in a general way, and also what we do, in a general way. For example, "mathematics" is a word which is used for a lot of activities which don't have much to do with each other. I am sure that the word means very different things for different people. For instance, you, Madam [*Serge Lang points to a lady in the audience*], what does "mathematics" mean to you?

LADY. The abstraction of numbers, the manipulation of numbers.

SERGE LANG. In fact, one can do mathematics without using numbers at all; as in geometry, or spatial mathematics. It's true that to give you an example of mathematics, as I shall do a little later, I shall use numbers, but in a context which, I think, will be different from the one you are thinking about. And you, sir, what does it mean, "mathematics"?

GENTLEMAN. The manipulation of structures.

SERGE LANG. Yes, but which ones? There are lots of structures which are not mathematical. Mathematics is not just a question of structures. For example, when you do physics, you also manipulate certain structures. In fact, the word "mathematics" is used in many different contexts. You have mathematics as they are done in elementary or high school. You have computer mathematics, applied to problems of communications. If you are into physics or chemistry, you use mathematics to describe the empirical world. But what I want to talk about today is what I will call "pure mathematics", those which are done from a purely aesthetic point of view. To do mathematics like that is very different from studying the empirical world. It's different from describing or classifying the empirical world by means of mathematical models. An experimental scientist makes a choice among many possible models, to find those which fit the empirical world, the world of experiments, in trying to find a system for the world. There are lots of pure mathematics which are not used in studying the empirical world, and which are considered solely for their beauty. And this has been the case forever, for centuries, since there have been civilizations–Arabic, Hindu, whatever. The Greeks did mathematics for the beauty of it.[1]

It is true that some parts of mathematics have their source in the empirical world, but much mathematics is done independently of these sources. This point of view has been expressed by other mathematicians,

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[1] Which does not exclude that they also did mathematics which had practical applications. Everyone agrees to include physics, chemistry, biology, under the general heading of "science". To decide whether "pure mathematics" as I have described them should also be placed under this heading is a question of terminology which I don't want to get into now.

and I want to read you something written by other mathematicians, for instance on the relation between doing mathematics as they relate to applied math.

Jacobi, who was a mathematician of the 19th century, wrote in a letter to Legendre:[2]

> I read with pleasure Mr. Poisson's report on my work, and think I can be very satisfied by it . . . but Mr. Poisson should not have reproduced a rather clumsy phrase by Mr. Fourier, who reproached Abel and me for not having preferred to work on heat flow. It is true that Mr. Fourier thought that the principal goal of mathematics was their public utility and their use in explaining natural phenomena. A philosopher like him should have known that the only goal of Science is the honor of the human spirit, and that as such, a question in number theory is worth a question concerning the system of the world.

In an article which appeared in the collection "Great Currents of Mathematical Thought", directed by F. Le Lionnais in 1948, Andre Weil (who is one of the great mathematicians of this century), quoted Jacobi in the following context:

> But if, like Panurge, we ask the oracle questions which are too indiscreet, then the oracle will answer as to Panurge: "Drink!" Advice which the mathematician is only too glad to follow, satisfied that he is to quench his thirst at the very sources of knowledge, satisfied that these sources always gush pure and abundant, while others must have recourse to the muddy paths of a sordid actuality. That if one reproaches him for his arrogant attitude, if one challenges him to engage himself in the actual world, if one asks why he persists on these high glaciers where none but others like him can follow him, he answers with Jacobi: "For the honor of the human spirit!"[3]

OK, that's literature. It's also a pompous style, which does not reflect accurately Jacobi's thoughts. To refer to others, who "must have recourse to the muddy paths of a sordid actuality" is not exactly the same thing as to say that "a question of number theory is worth a question concerning the system of the world". Weil, elsewhere, described in another way his

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[2] No date, stamped 2 July 1830, *Collected Works of Jacobi,* Vol. 1, p. 454.

[3] The original is in French, and very literary French at that:

> Mais si, comme Panurge, nous posons a l'oracle des questions trop indiscretes, l'Oracle nous répondra comme à Panurge: Trinck! Conseil auquel le mathématicien obéit volontiers, satisfait qu'il est de croire étancher sa soif aux sources memes du savoir, satisfait qu'elles jaillissent toujours aussi pures et abondantes, alors que d'autres doivent recourir aux sentiers boueux d'une actualité sordide. Que si on lui fait reproche de la superbe de son attitude, si on le somme de s'engager, si on demande pourquoi il s'obstine en ces hauts glaciers ou nul que ses congénères ne peut le suivre, il répond avec Jacobi: "Pour l'honneur de l'esprit humain!"

own reasons to do mathematics. In an interview published in "Pour la Science" (November 1979, the French version of "Scientific American") he says:

> According to Plutarch, it is a noble ideal to work to make one's name immortal. Ever since I was young, I hoped that my work would have a certain place in the history of mathematics. Is that not a motivation as noble as to try to get a Nobel prize?[4]

So, it's not so much for the honor of the human spirit, it's for the honor of his own spirit. I think rather that one does mathematics because one likes to do this sort of thing, and also, much more naturally, because when you have a talent for something, usually you don't have any talent for something else, and you do whatever you have talent for, if you are lucky enough to have it. I must also add that I do mathematics also because it is difficult, and it is a very beautiful challenge for the mind. I do mathematics to prove to myself that I am capable of meeting this challenge, and win it.

So one does mathematics, but that does not mean people are unhappy if the mathematics they do is sufficiently good to make it in the history books. Of course, all the mathematicians that I know are perfectly happy when they do mathematics at this level. They are happy with the possible honors they may get from it, and they are happy to leave a name in mathematics. But I would not say that they do mathematics specifically for this purpose, that they give themselves to mathematics, whether they be pure or applied.

If I ask you what music means to you, would you answer: "The manipulation of notes"? When one does pure mathematics, one does something quite different from "manipulating". To make clear the reasons behind people doing pure mathematics, from an aesthetic point of view, I have to give you an example. But to show you what mathematics is, if you are not yourself in mathematics, I have difficulties which are analogous to those which I would have if I tried to tell an ancient Japanese, or a Hindu who never had contact with Western civilization, what a Beethoven symphony or a Chopin ballade is like. If you take someone totally foreign to Western culture, and deaf besides, how can you make that person realize what a Beethoven symphony or a Chopin ballade is like? It's impossible. Even if the person is not deaf, and is able to listen, it is still almost impossible if the person has no connection with Western culture, if the person has not heard these pieces several times. Western music is too different from Japanese music, or Hindu music; it is played on different instru-

[4] In a conference at the International Congress of Mathematics in Helsinki, 1976, reproduced in his *Collected Works* Vol. III, Weil had already touched this theme: "That mankind should be spurred on by the prospect of eternal fame to ever higher achievements is of course a classical theme, inherited from antiquity; we seem to have become less sensitive to it than our forefathers were, although it has perhaps not quite spent its force."

ments, with different orchestrations, with different rhythms, etc. So there is a great difficulty in making somebody understand what it's about. And conversely, Koto or Sitar concerts here in Paris don't happen so often, and affect only a small number of people.

Besides, there is a difficulty which occurs in all aesthetic situations: somebody may like one thing and not another. There are people who like Brahms and don't like Bach; who like Bach and don't like Chopin; who like Chopin and don't like Dowland (an English composer of lute pieces and lute songs at the time of Shakespeare).

How are you going to make somebody understand what a song by Dowland is like, or a Chopin ballade, without making them listen? It's impossible! And it's much easier to make you listen to some music than to make you do mathematics, because to listen to music you are in a passive state. You are taken in by the musical aesthetic, and you let the composer and interpreter take the active part. But to do mathematics, you need a much higher degree of concentration, and a personal effort. Furthermore, to make you do mathematics, I have to find a topic which is sufficiently deep, which is a real topic of mathematics, recognized as such by mathematicians. I can't cheat, but still I have to be able to explain things with words which everybody will understand. There are only very few such topics; and since I have to make a choice, maybe some people will like it and some others won't like it.

The topic has to be sufficiently deep to make you understand why some people will do mathematics all their life, and perhaps will neglect their wives, or husbands, or children, or God knows what. By the way, let me read you two sentences from a letter by Legendre to Jacobi[5] who had just gotten married rather late in life:

> Congratulations for having met a young wife who, after a *rather long* experience, you decided will make you happy forever. You were of a suitable age to get married. A man destined to spend a lot of time working in his office needs a companion who will deal with all the details of housework, and saves her husband from having to worry about those small day to day items which a man is not able to handle.

The sentence has a funny ring, especially in our "liberated" age.

Well, I have been talking in generalities for about ten minutes, that's enough. Now let's do mathematics. In the choice of the subject, I am very restricted, and it was almost necessary to pick a topic having to do with numbers. It concerns prime numbers.

Who has heard of prime numbers? [*Varied reactions and response in the audience.*] Almost everybody, or nobody? Raise your hand. Who has never heard of prime numbers? [*Almost everybody in the audience has heard about prime numbers and knows approximately what the word means.*] For instance, you, Madam, what are the prime numbers?

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[5] Written 30 June 1832, loc. cit p. 460.

LADY. 1, 3, 5, 7 . . .

SERGE LANG. No! These are the odd numbers. I mean the prime numbers, that is 2, 3, 5, 7, 11, 13. What's the next one?

LADY. 17, 19 . . .

SERGE LANG. Very good, you have understood what a prime number is.

LADY. I forgot 2.

SERGE LANG. Yes, you are right. I misunderstood. But it is a general convention that 1 is not called a prime number. So to say that a number is prime means that it is at least equal to 2, and that it is divisible only by itself and by 1.

The number 4 is not prime because $4 = 2 \times 2$.
6 is not prime because $6 = 2 \times 3$.
8 is not prime because $8 = 2 \times 4$.
9 is not prime because $9 = 3 \times 3$.

And so on. As for the prime numbers, we have already listed them up to 19. After that, we find 23, 29, 31, 37 . . .

Now here is a question about prime numbers. Are there infinitely many of them or is there only a finite number of them?

LADY. Yes, infinitely many.

SERGE LANG. Very good. How do you prove it?

LADY. I don't know.

SERGE LANG. [*Pointing to a young man.*] You, do you know how to prove it?

YOUNG MAN. Mathematicians have found millions of them.

SERGE LANG. No, I don't mean finding millions of them, I mean prove that the sequence of prime numbers does not stop.

[*Brouhaha in the audience, various proofs are suggested by some people.*]

SERGE LANG. Are you a mathematician? Yes? OK, I ask the mathematicians in the audience not to say anything. I am not talking here for them. [*Laughter.*] Otherwise, it's cheating.

I say that there are infinitely many prime numbers. This means that the sequence of prime numbers does not stop. And I am going to prove it, because there is a very simple proof, which is also very old, and is attributed to Euclid. Here is how the Greeks did it.

Let's start with a remark. Take any integer, that is a whole number, for instance 38, which I can write as 2×19 where 2 and 19 are prime numbers. Then 38 is a product of these two prime numbers. If I take 144, then I can write

$$144 = 12 \times 12 = 3 \cdot 4 \cdot 3 \cdot 4 = 3 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 2.$$

Again, it's a product of prime numbers, and I have written some of them several times. In any case, I can always express an integer as a product of prime numbers. Because if I am given an integer N bigger than 2, then either N is already prime or N can be expressed as a product of two smaller numbers. Each one of these smaller numbers is either prime, or can be expressed as a product of still smaller numbers. If you continue this process, you end up with prime numbers.

Now let's give the Greeks' proof that there are infinitely many primes. We are going to see that if we make a list of the primes

$$2, 3, 5, 7, 11, 13, 17, \ldots, P$$

going from 2 to P, then we can always find another prime number which is not in this list. We proceed as follows. I take the product of all the primes in the list. This gives me some number, to which I add 1. Let N be this new number. Thus we have

$$N = (2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot P) + 1.$$

Then either N is prime, or N is not prime. If N is prime, it is not equal to any of the ones we listed from 2 to P, and so we have just constructed a new prime number. If N is not prime, then we can express N as a product of primes. In particular, we can write $N = qN'$, where q is a prime number dividing N. Can q be equal to any one of the primes from 2 to P?

PEOPLE IN THE AUDIENCE. It's a new one.

SERGE LANG. Why? Let's pick on somebody. You, the young man over there.

YOUNG MAN. For the others, the division does not come out exact.

SERGE LANG. That's right, if we divide N by q then there is no remainder; but if we divide N by one of the primes between 2 and P there is a remainder of 1. So we discovered a new prime which was not in the list. This means that you can't make up a finite list of all the prime numbers, and this concludes the proof.

Now how are the primes distributed among all numbers? Is there some rule which tells you how many there are? How they are distributed among all integers?

A GENTLEMAN. There are millions of them.

SERGE LANG. Sure, there are also billions, but that's not the question I am raising. For example, how many primes are there smaller than 10,000, approximately? Can you answer that?

SOMEONE. You can count them.

SERGE LANG. That's true, but if I said up to 1,000,000, or up to an arbitrary number x? Let's put the question differently. Is there a formula which gives the number of primes less than x? Who says yes? An approximate formula. [*Hesitations in the audience, people comment simultaneously.*] OK, it's complicated. I would have to describe the primes more precisely. Let's not go into this right away. Let me go on to raise other types of questions about primes. In particular, what are called the twin primes.

For example:

3 and 5 differ by 2;
5 and 7 differ by 2;
11 and 13 differ by 2;
17 and 19 differ by 2;
29 and 31 also.

One says "twin primes" for obvious reasons.

Now, is there an infinite number of primes like that, an infinite number of twin primes?

Who says yes? Raise your hand. [*Some hands go up.*]

Who says no? [*Other hands go up.*]

Who keeps a prudent silence? [*Many hands go up. Smiles.*]

Who thinks it's an interesting question?

THE AUDIENCE. Yes, it's interesting. [*Several people talk at once.*]

SERGE LANG. Of course, you can like it or not like it. In fact, mathematicians generally think it's an interesting problem. Well, you see, it's a problem. No one knows the answer. If you find the answer, you will be like in Plutarch, you'll make it in the history of mathematics. In fact, one thinks there are an infinite number, and one can even do better than that. One can try to understand why there should exist an infinite number of twin primes.

SOMEONE. Is there an infinite number of triplets?

SERGE LANG. The question is interesting. Can you answer it right away?

SEVERAL VOICES IN THE AUDIENCE. Yes, I think there is an infinite number.

SERGE LANG. Watch out! Let's try to add a number to the couples of primes we already have.

3 5 7
5 7 9
11 13 15
17 19 21
29 31 33
etc.

SOMEONE. 21 is not prime.

SERGE LANG. Yes. What do you notice with your triplets? There is one, 3, 5, 7. But after that, what happens? You don't know? Look carefully: 9, 15, 21, 33 . . .

AUDIENCE. They are multiples of . . .

SERGE LANG. Shhh! The gentleman over there. [*Hesitations. No answer from the gentleman.*] They have a property, those numbers: they are all divisible by 3. That's a very easy exercise, to show that in every triplet of odd numbers, there is always a multiple of 3. Hence there cannot be a triplet of prime numbers.

AUDIENCE. Except the first one, 3, 5, 7.

SERGE LANG. Except the first, of course, which also has a multiple of 3, but 3 is prime, and there won't be any other.

Let's go back to the twin primes, the couples of primes if you want. Let's try to understand why there should be an infinite number of them. But before, let's go back to the question: how many primes are there less than or equal to x? An approximate formula.

OK, let's take all integers up to x:

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, x.$$

Among these numbers, you have the even numbers and the odd numbers. What does it mean that a number is prime? It means that it is divisible only by itself and 1. Therefore, if a number if prime, it is certainly not even.

AUDIENCE. Except 2.

SERGE LANG. Of course, except 2. Now, if I go up to x, how many odd numbers are there?

SEVERAL VOICES IN THE AUDIENCE. Half of them.

SERGE LANG. Approximately half. That's right, $x/2$. It's a certain fraction of x. The number of primes less than or equal to x will be a certain fraction times x. And this fraction will depend on x. It is this fraction which we are trying to determine.

All right, so among all the integers 1, 2, 3 up to x, there will be approximately half of them which will be odd, so not divisible by 2. Among the odd numbers, how many will not be divisible by 3?

AUDIENCE. One third.

SERGE LANG. No, one third is divisible by 3 and two thirds won't be divisible by 3. OK? Let's write $2/3$ in the form $(1 - 1/3)$. Now among the remaining ones, how many will not be divisible by 5?

A VOICE IN THE AUDIENCE. $1 - 1/5$.

SERGE LANG. Are you a mathematician? Yes? Then shut up! It's cheating. It's not nice. Among the remaining ones, how many are there which are not divisible by the next prime number?

AUDIENCE. $1 - 1/7$.

SERGE LANG. Good, and then finally to find the numbers which are prime, what do we need? We need that they should not be divisible by any prime number, from 2 to . . . somewhere. Thus we are led to take the product

$$\frac{1}{2}(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})\cdots$$

which must go up to where?

AUDIENCE. Up to the last prime number before x.

SERGE LANG. Yes, but one can do better than that. Anyhow, at worst, it will be the product

$$\text{product of all factors } (1 - \frac{1}{p})$$

where p goes up to x. This will be approximately the fraction of x which gives the fraction of all numbers which are prime.

Now, in fact, I don't need to go up to x. I need to go only up to the square root of x, which is denoted by $\sqrt{x}$. Because suppose that a number which is smaller than x and is not prime, is divisible by some prime bigger than $\sqrt{x}$. Then it is necessarily divisible by a prime number smaller than $\sqrt{x}$.[6] Hence we can eliminate such a number when we have met the smallest of its prime factors. But when x is large, and when p is between $\sqrt{x}$ and x, the term $(1 - 1/p)$ is very close to 1. One can show that the product taken over all p with $\sqrt{x} \leqq p \leqq x$ is close to 1/2. To simplify the formulas, I shall continue to write the product with $(1 - 1/p)$ for all $p \leqq x$. To have a better approximation, or the best possible approximation, I would anyhow have to multiply the product by a constant which is hard to determine, and which reflects relations which are more hidden than the relation which we have just described.

Here I count approximately, and I am led to consider that product. It gives approximately the fraction of x giving the number of primes less than or equal to x. This fraction of x is rather mysterious, but still, it gives some idea of what's happening. For instance, is this fraction constant? Clearly not. The further we go, the smaller it becomes. If I take x very large, the fraction will be small. How fast it becomes small is not clear. It's not at all clear how this product behaves. And now, I am stuck. I will give you some answer later, but I won't be able to prove it because it would get too technical.

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[6] I give the details of this assertion. Let N be less than or equal to x. Suppose that N is a product, $N = pN'$ with p prime greater than $\sqrt{x}$. Then $N' = N/p$, and N' is smaller than $\sqrt{x}$. If q is a prime factor of N', then q is smaller than $\sqrt{x}$ and is also a factor of N.

It's complicated to analyze this product, but still, we have made a step forward by finding this product, which gives us some fraction of x, decreasing as x increases.

Mathematicians use the sign

$$\prod$$

to denote a product. So we denote the product of all the factors $(1 - 1/p)$, taken for all primes p less than or equal to x, by the symbol

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right).$$

The number of primes $\leqq x$ should then be approximately equal to

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right) \cdot x.$$

Since it's a little heavy to write the product, we are going to express it by a single letter, $F(x)$ (F for "fraction", depending on x). So we let

$$F(x) = \prod_{p \leqq x} \left(1 - \frac{1}{p}\right).$$

With this abbreviation, we can then write that the number of primes $\leqq x$ is approximately equal to

$$F(x)x,$$

which looks simpler.

Now, let's try to apply the same analysis to the twin primes. What happens to twin primes which did not happen for all prime numbers? There is one extra restriction: if p is prime, then $p + 2$ must also be prime. Let's take all numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \text{ up to } x.$$

About half of these are odd. So again we get a factor of $1/2$. Now let's look at those which are not divisible by 3, and let us write under each number the remainder after we divide it by 3:

$$\begin{matrix} 1 & 2 & 3 & \quad 4 & 5 & 6 & \quad 7 & 8 & 9\ldots \\ 1 & 2 & 0 & \quad 1 & 2 & 0 & \quad 1 & 2 & 0\ldots \end{matrix}$$

Since p cannot be divisible by 3, after we divide it by 3 we get a remainder of 1 or 2. We have two possible choices.

For the twin primes, both p and $p + 2$ must be prime. So not only p is not divisible by 3 but also $p + 2$ is not divisible by 3. This means that when we divide p by 3, the remainder must be . . .

THE AUDIENCE. Different from 1.

SERGE LANG. Yes, because if the remainder is equal to 1, and if I add 2, then $p + 2$ is divisible by 3. So we have found a new condition on p, that after dividing by 3, the remainder must be 2. Instead of excluding just one possibility, as we did before, we now exclude two possibilities. Our product therefore starts with

$$\frac{1}{2}(1 - \frac{2}{3}).$$

Now let's do the same thing with 5. If we divide an integer by 5, and the integer is not divisible by 5 exactly, then there are four possible remainders, namely 1, 2, 3, 4. Among these, if I add 2, I want that the number $p + 2$ is also not divisible by 5. Then how many possible remainders are there? In other words, in order that $p + 2$ is not divisible by 5, the remainder should not be equal to what?

AUDIENCE. 3.

SERGE LANG. Yes indeed, if we divide the integer by 5, the remainder should be different from 0 or 3. This gives me a factor

$$\frac{3}{5} \quad \text{or} \quad (1 - \frac{2}{5}).$$

Next for 7, I want to characterize those integers p which are not divisible by 7, and such that, if I add 2, then $p + 2$ is not divisible by 7. Then I must omit multiples of 7, and in addition those whose remainder after dividing by 7 is equal to 5. The next factor will therefore be . . .

AUDIENCE. $(1 - 2/7)$.

SERGE LANG. Excellent. Therefore the fraction we are looking for will be the product

$$\frac{1}{2} \prod (1 - \frac{2}{p}),$$

taken over all prime numbers $\geqq 3$ and less than or equal to x. When we considered all prime numbers, without any further restriction, we were led to take the product of all terms $(1 - 1/p)$. Now with the extra condition that $p + 2$ is prime, we are led to the product of the terms $(1 - 2/p)$. All of this is approximate, but it gives a good idea about how many twin primes there are. That's the conjecture:

Conjecture. The number of twin primes less than or equal to x is approximately equal to

$$\frac{1}{2} \prod_{3 \leqq p \leqq x} (1 - \frac{2}{p})\, x.$$

Here again, the product changes with x, it is a function of x. It is not a constant function like 4/5, or 1/12. As before, we abbreviate the product, and we let

$$F_2(x) = \frac{1}{2} \prod_{3 \leqq p \leqq x} (1 - \frac{2}{p}),$$

so that the number of twin primes $\leqq x$ is approximately equal to $F_2(x)x$. We are now in a similar situation as when we were counting all the prime numbers, and there remains to analyze this product, which is taken over prime numbers even though we are trying to count prime numbers. There is something a little circular here, but not completely.

We get some information from this product. One can compute this product. Even though the fraction

$$\frac{1}{2} \prod_{3 \leqq p \leqq x} (1 - \frac{2}{p})$$

decreases with x, this fraction is still rather large, but I would have to explain what I mean by "rather large". Now I am stuck, one can say it only with some more advanced vocabulary, with slightly more knowledge of mathematics. Up to now, I could manage only with the basic rules of arithmetic that one uses in the 7th grade. But let's try anyhow.

Who has heard of the logarithm? [*A few hands go up.*] Who never heard of the logarithm? [*A few hands go up.*] Who keeps a prudent silence? [*Several hands go up.*] OK, there is something that's called the logarithm. It is denoted by $\log x$. You will find it on all the little hand calculators in the drug stores. I don't have time to explain it in greater detail. [*A few more explanations are given later.*]

Then it is true that

$$\prod_{p \leqq x} (1 - \frac{1}{p}) \text{ is approximately equal to } \frac{1}{\log x}.$$

But this is not trivial to prove, and there is no way I can give you any idea how it is proved. It's quite technical, and it's even tough to do. It's elementary if you start from differential and integral calculus, but even being elementary, it's tough. You might manage in say . . . thirty pages.

[*Various reactions in the audience.*]

SERGE LANG. Oh, you know, thirty pages, it's nothing. Six months ago some new theorems got proved that required 10,000 pages. So thirty pages, it's no big deal. Starting from scratch, of course.

Anyhow, there is a function which is called $\log x$, and the first product

$$\prod_{p \leqq x} (1 - \frac{1}{p})$$

is approximately equal to $1/\log x$.

As for the other product, associated with the twin primes, one can prove that

$$F_2(x) \text{ is approximately equal to } \frac{1}{(\log x)^2}.$$

The square comes from the fact that we replace $1/p$ by $2/p$. For example, we have

$$(1 - \frac{1}{p})^2 = 1 - \frac{2}{p} + \frac{1}{p^2},$$

and if p is large, then $1/p^2$ is very small compared to $2/p$. So approximately we can leave it out, and we find that

$$\prod (1 - \frac{1}{p})^2 \text{ is approximately equal to } \prod (1 - \frac{2}{p}).$$

Therefore, the conjecture is:

The number of twin primes less than or equal to x is approximately equal to

$$F_2(x)x, \quad \textbf{or also to } \frac{x}{(\log x)^2}.$$

Naturally, I would still have to explain more precisely what I mean by "approximately", and I don't have the time now to do so. It is a little more technical. Maybe we will have time later, after the talk.

The function $\log x$ is a function which grows slowly with x. Therefore our fraction is relatively large. But in spite of these heuristic arguments, nobody knows how to prove that there exists an infinite number of twin primes.

What have I just done? There is no doubt that we have been doing mathematics! But nothing has been proved, except the first theorem of Euclid. We have given arguments which were only heuristic, but that does not mean that the mind did not function. On the contrary. We formulated a conjecture, which means that we tried to guess what was the answer, and we now face a problem. Well, that's what it means to do mathematics: find interesting problems and try to solve them. Eventually, solve them.

Now let's raise another question. We observe that:

$2^2 + 1 = 4 + 1 = 5$ is prime
$4^2 + 1 = 16 + 1 = 17$ is prime
$6^2 + 1 = 36 + 1 = 37$ is prime
$8^2 + 1 = 64 + 1 = 65$ is not prime
$10^2 + 1 = 101$ looks like it should be prime; in fact it is prime.

Question: In this list of prime numbers which can be written as the square of a number plus one, are there infinitely many primes? Think about it, I am asking for your intuition. I am not asking you to prove anything yet. Are there infinitely many primes of the form $n^2 + 1$?

SOMEBODY. No.

SERGE LANG. Who says yes . . . ? Who says no . . . ? Who keeps a prudent silence? [*Varied reactions in the audience. Guesses go both ways.*] It's less clear, isn't it?

AUDIENCE. There is more space between them. They occur less frequently.

SERGE LANG. That's right, madam, there is more space between them. And there is more space than there was between the twin primes, which in turn had more space between them than all the primes. Can we guess how much space there should be, approximately? A little? A lot? Can you give a quantitative measure?

First let me give you the answer: nobody knows if there exists an infinite number. It's an unsolved problem. It's one of the great problems of mathematics. One thinks that the answer is yes. I repeat, if you find the answer, you will make it into the history books of mathematics (but you didn't necessarily do it with that purpose in mind).

The conjecture is that there exists an infinite number of primes of the form $n^2 + 1$, but like for the twin primes, one can do better than that. We can give some idea of the corresponding fraction that they represent.

For all the primes, the fraction is

$$F(x)x \quad \text{or} \quad \frac{1}{\log x}\, x = \frac{x}{\log x}.$$

For the twin primes, the fraction is

$$F_2(x)x \quad \text{or} \quad \frac{1}{(\log x)^2}\, x = \frac{x}{(\log x)^2}.$$

What fraction are we going to find for the primes of the form $n^2 + 1$?

SOMEONE. You must necessarily have n smaller than $\sqrt{x}$.

SERGE LANG. Right! If $n^2 + 1$ is smaller than x then n is bounded by $\sqrt{x}$. Let's try to guess what fraction of all numbers is represented by the primes of the form $n^2 + 1$. If the primes are distributed at random, it is probably the same fraction of $\sqrt{x}$ as the fraction of all primes with respect to x. It's rather plausible. Anyway, it's a working hypothesis. So what is the conjecture? The gentleman over there.

GENTLEMAN AND THE AUDIENCE. [*Everyone hesitates.*]

SERGE LANG. The fraction of those primes less than or equal to x is

$$\frac{1}{\log x}\, x.$$

If you apply this to $\sqrt{x}$ you get approximately

$$\frac{1}{\log x}\sqrt{x}.$$

That's the conjecture, roughly speaking, up to a constant factor.

SOMEONE. Why not $\dfrac{1}{\log \sqrt{x}}\sqrt{x}$?

SERGE LANG. OK, it's not so clear if it should be x or $\sqrt{x}$. But first, one has the relation

$$\log \sqrt{x} = \frac{1}{2}\log x,$$

so the two expressions differ only by a factor of 2; and second, I don't claim to give anything but an approximation, up to some constant factor. In any case, these heuristic ideas, which are purely intuitive, give you the idea that there should be an infinite number of such primes, since one can give a quantitative measure for them.

Of course, I should explain what I mean by "approximately", not only for the primes of the form $n^2 + 1$, but also for all the primes, or the twin primes. This would be the topic for another talk, which I can't give today and which would last perhaps one hour. It's precisely the error term in this approximation that is the subject of a problem which is generally recognized as being the greatest problem in mathematics. It's the error term which appears in the formula $x/(\log x)$ for all the prime numbers. There is a precise conjecture, due to Riemann, and called the Riemann hypothesis, made about 130 years ago, and which gives the best possible error term. It's still not proved today, despite the fact that many mathematicians have worked on it.

But I have been talking for an hour. Let's stop here.

The questions

QUESTION. You have mentioned other pure mathematicians, but you, why do you do this kind of work?

SERGE LANG. Why? Why do you compose a symphony or a ballad? I already told you why. Because it gives me chills in the spine. That's why. But I did not say you should also get them. That's freedom for you.

QUESTION. Can you say where is the limit between pure and applied mathematics?

SERGE LANG. There are no limits. The two mix with each other without my being able to define a limit. If you try to define a limit more precisely, in general, I don't say you won't succeed, but I have personally never seen anyone succeed in doing so.

QUESTION. What you did just now, do you think it could be useful somewhere?

SERGE LANG. You said "could". This is a conditional, so I am forced to answer logically: yes.

QUESTION. When you do mathematical research, do you have a goal in mind?

SERGE LANG. The goal is to prove the conjecture.

QUESTION. But at the start?

SERGE LANG. At the start, it's first to find the conjecture that you want to prove, and then try to prove it. One of the main difficulties in mathematics is to find the subject on which you want to concentrate, and the problem which you are going to try to solve.

QUESTION. But is that done by logical deduction or intuition?

SERGE LANG. Have I done any logic here? Half and half. There was a lot of intuitive stuff, and logic, you know, when I tell you that something or other is one third or one fifth of something else, I have assumed a lot of things without proving them. It's more by intuition than by logic that I have been doing mathematics here. Anyway, in general, new results are discovered by intuition, proofs are discovered by intuition, and finally they are written up according to a logical pattern. But don't confuse the two. It's the same as in literature: grammar and syntax are not literature. When you write a musical piece, you use notes, but the notes are not the music. To read a piece of music from the written text is not a substitute for hearing the piece in Carnegie Hall or elsewhere. Logic is the hygiene of mathematics, just as grammar and syntax are the hygiene of language–and even then! "Under the bam, under the boo, under the bamboo tree . . . ", there isn't any grammar. The essential thing in Shakespeare, or Goethe, is not grammar or syntax. It is the poetry, the musical effect of words, poetic allusions, aesthetic impressionism, and many other things. But whereas the beauty of poetry pales under translation, the beauty of mathematics is invariant under linguistic transformations.

QUESTION. You have used heuristic arguments, and approximations to describe what a pure mathematician does. But a mathematician does other things besides that.

SERGE LANG. Watch out, I did not say that a mathematician does only that. One tries to prove something, one discovers a conjecture a little like I have described here. But once the conjecture has been made, one tries to prove it. Sometimes we succeed, sometimes we don't. We proceed by successive approximations, both in making guesses and trying to prove them. The negation of one absolute is not the absolute of opposite type.

Depending on how often you succeed, or how deep are your results, you will be a great mathematician, or an average one, or . . .

QUESTION. For instance, you haven't talked about axiomatization.

SERGE LANG. Axiomatization is what one does last, it's rubbish. It's the hygiene of mathematics, axiomatization. It's the discipline of the mind. Like grammar and syntax. But do what you want. Each one has to determine what they like to do. The word "rubbish" is too strong. I also axiomatize, when I find it appropriate to do so, and there are lots of other things I have not talked about. I made a choice. I wanted to show an essential aspect of mathematics which most people have no idea exists.

SOMEONE. There is a problem that gives me chills in the spine, the problem of the denumerability of the real numbers. Cantor tried to deal with this problem, and I think he became a little crazy because of it. I have heard that Cantor proved it. I'd like to know if this is true.

SERGE LANG. Proved what? that the real numbers are not denumerable? Yes, he surely did.

SAME. Can you give us an idea of the proof?

SERGE LANG. [*Hesitates.*]

SAME. Without going too far.

SERGE LANG. OK, the gentleman would like . . . [*Brouhaha in the audience.*]

Yes! I can do it in just a few minutes.

GENTLEMAN. I was just curious.

SERGE LANG. But that's all it ever is, curiosity! [*Laughter.*] On the contrary, the whole point of the operation was to sharpen your curiosity by showing you what I was curious about. So I give the proof. What is a real number? It's an infinite decimal, for example 27.9130523 . . . Since I can't write an infinite number of digits like that, I have to use some notation with indices. And to simplify matters, I'll consider only the numbers between 0 and 1. Suppose that we can write all these numbers in a sequence, with a first, a second, a third, and so on, without missing any of them, as follows:

$$0.a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots$$

$$0.a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots$$

$$0.a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots$$

with integers a_{ij} between 0 and 9. I am going to show that there is some infinite decimal which is not in this list. I choose an integer b_1 which is not equal to a_{11}. Then an integer b_2 which is not equal to a_{22}. Then an integer b_3 which is not equal to a_{33}. In general, I choose an integer b_n which is not equal to a_{nn}, and I pick b_n between 1 and 8 (to avoid ambiguities having to do with a sequence of 0's or 9's). Then the infinite decimal

$$0.b_1\, b_2\, b_3\, b_4 \ldots$$

is not equal to any decimal in the list because of the way I have constructed it, so it is a new one.

Note that what we have just done is similar to Euclid's method at the beginning. We made a list, and then we showed that there is a decimal which is not in the list.

QUESTION. I would like to know what you think of the great schools of mathematical thought concerning infinity.

SERGE LANG. I don't think about it. All of this was settled for me long ago. It had some historical importance, but today, it's settled. Something is either infinite or it is not.

QUESTION. But it's not as simple as that!

SERGE LANG. OK, you are right.

QUESTION. Does infinity exist?

SERGE LANG. When I mentioned prime numbers, did you know how to answer whether there was an infinite number of them or not?

QUESTION. Yes.

SERGE LANG. Then that's it, you have understood. That settles the question.

QUESTION. But Cantor's proof was more or less rejected by the intuitionists. I think there was a lot of fighting about this subject.

SERGE LANG. If people want to fight, they are free to do so. I just do mathematics.

QUESTION. Have you worked yourself on the problems you raised today?

SERGE LANG. Yes, on the problem of primes of the form $n^2 + 1$. Since that interests you, and you are still sitting here, let me give a few more precise statements about that problem. When I started to think about what I would tell you today, I thought of the twin primes but I did not know myself if there was a conjecture about them, nor how to motivate it. I looked up the book by Hardy and Wright, and I found it. This conjecture, and the one about the primes of the form $n^2 + 1$ are due to Hardy and Littlewood, in an article dating back to 1923. I am going to state their conjecture somewhat more precisely than I have done so far.

I have said several times that certain expressions were approximate, up to a constant factor. What does this mean? Suppose I have two expressions $A(x)$ and $B(x)$. We say that $A(x)$ is asymptotic to $B(x)$ if the quotient

$$\frac{A(x)}{B(x)}$$

approaches 1 when x grows larger and larger. This means that when x is very large, then the quotient is very close to 1. The relations that $A(x)$ is asymptotic to $B(x)$ is denoted by the symbol

$$A(x) \sim B(x).$$

We can then state the prime number theorem as follows.

Let $\pi(x)$ be the number of primes $\leqq x$. Then we have the relation

$$\pi(x) \sim e^{\gamma} F(x) x,$$

where e and γ are constants used all the time in mathematics and F is as before. The constant e is called the natural base for logarithms; and γ is called Euler's constant. Since the product $F(x)$ itself looks rather mysterious, one prefers to replace it by another expression. It is a theorem due to Mertens that one has the asymptotic relation

$$e^{\gamma} F(x) \sim \frac{1}{\log x},$$

and therefore we find that

$$\pi(x) \sim \frac{x}{\log x},$$

which is the usual formulation for the prime number theorem. It is useful to write it this way, because the log function is very well known. We know how it grows when x becomes large. For instance, we have the following values:

$\log 10 = 2.3\ldots$	$\log 10{,}000 = 9.2\ldots$
$\log 100 = 4.6\ldots$	$\log 100{,}000 = 11.5\ldots$
$\log 1000 = 6.9\ldots$	$\log 1{,}000{,}000 = 13.8\ldots$

and so on. Observe that the numbers 10, 100, 1,000, 10,000, 100,000, 1,000,000 grow by powers of 10, but the logarithm grows only by adding approximately 2.3 each time. This means that the logarithm grows much more slowly.

Similarly, let $\pi_2(x)$ denote the number of twin primes $\leqq x$. Then Hardy–Littlewood's conjecture is that

$$\pi_2(x) \sim (e^{\gamma})^2 F_2(x) x.$$

This formula can also be written asymptotically with the logarithm, in the form

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2},$$

where C_2 is a constant, given by an infinite product taken over all primes $\geqq 3$, namely

$$C_2 = \prod_{3 \leqq p} \left[1 - \frac{1}{(p-1)^2} \right].$$

Hardy and Littlewood give probabilistic arguments more precise than those I could give here in one hour. In particular, when I wrote the products, I was assuming implicitly that the conditions of divisibility by 2, 3, 5, etc. were independent. But I did not prove this assumption, which in fact is false. These conditions are not independent, and the constant e^{γ} reflects the dependencies between these divisibility conditions.[7] But this is now getting much more technical, and I cannot go into the details necessary to find the constant e^{γ}. I have to refer you to the original article by Hardy–Littlewood, or the book by Hardy and Wright.

To come back to the question about my own work, I and a friend Hale Trotter have been interested in analogous problems, concerning the distribution of prime numbers in much more complicated contexts. I can't go into them here. But we rediscovered the same asymptotic relation as Hardy–Littlewood for the primes of the form $n^2 + 1$, with the same constant C_2 (fortunately!). The article with Trotter gives a probabilistic model which is completely different from that of Hardy–Littlewood. Naturally, only someone who has specialized in number theory can understand it.

QUESTION. Between pure and applied mathematics, I don't see the difference very well.

SERGE LANG. At first sight, to compute the number of primes of the form $n^2 + 1$ has no applications. This does not mean that it will never have applications. In the history of mathematics, the results of research done purely from an aesthetic point of view have been applied, sometimes after a century, to very concrete problems. For instance, today, one uses parts of the theory of prime numbers in coding theory. As far as I know, it's not the same theorems that we have discussed today, but it could very well be.

I have also brought a quote from von Neumann,[8] which I did not have the time to read before. Maybe it's time to read it now. [*Approval from the audience.*] OK, here it is.

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[7] The proof of the conjectured formula for the number of primes is not at all trivial. Indeed, the Goldbach problem, which is entirely analogous to the twin prime problem, states that every sufficiently large even number is the sum of two odd primes. Hardy and Littlewood have even conjectured that there is an asymptotic formula for the number of such representations, given by

$$N_2(n) \sim 2C_2 \frac{n}{(\log n)^2} \prod \frac{p-1}{p-2},$$

where the product (finite) is taken over all the primes $\neq 2$ dividing n. Note again the same constant C_2 which we found in the twin prime problem, as well as the denominator with the square of the logarithm. The heuristic arguments are similar. But Hardy–Littlewood remark that Sylvester in 1871 and Brun in 1915 had conjectured a false formula, which did not take into account the relations giving rise to the factor e^{γ}.

[8] J. von Neumann, *The Mathematician*, Collected Works I, pp. 1–9.

> I think it is a relatively good approximation to truth—which is much too complicated to allow anything but approximations—that mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure. But, once they are so conceived, the subject begins to live a peculiar life of its own and is better compared to a creative one, governed almost entirely by aesthetical motivations, than to anything else, and in particular, to an empirical science. There is, however, a further point which, I believe, needs stressing. As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality," it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art.* This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up. It would be easy to give examples, to trace specific evolutions into the baroque and the very high baroque, but this, again, would be too technical.

I have some objections to the way von Neumann expresses himself. If he expresses merely his personal tastes, well and good. He has the right to his own tastes. Unlike him, I don't feel any danger about doing mathematics for which I see no relation with the empirical world. Many times during the course of my life, I have seen situations when some mathematicians complained that certain fields of research were too "abstract"—von Neumann might say "baroque". But fifteen years later, such research combined with other led to the solution of very classical problems, which had been raised already in the 19th century.

There are as many possibilities to do uninteresting or trivial mathematics in number theory as there are doing mathematics with empirical connections. As for "inbreeding", I don't understand what von Neumann means. Many of the most beautiful discoveries in mathematics come from the wedding of branches which a priori seem very far apart from each other. One of the characteristics of mathematical genius is the ability to bring together different branches, by what could be called "inbreeding", or to bring together threads going off into many directions; to find fundamental ideas in the mass of details and complexities which others have accumulated. This does not mean that the work of others has been worthless.

Historically, in the 50's, it is true that several branches of pure mathematics developed parallel to each other. Von Neumann was not the only one to complain that these streams, which for many at the time seemed without connection to each other, were too abstract. But in the 60's, we have seen these streams come together in some very deep and essential ways. And not only that, but we have seen them join with subjects which had not been fashionable for forty years, and we have seen them join with subjects which had been almost forgotten since the 19th century. We have also seen old conjectures proved precisely because in the last fifteen years, people have found how to make syntheses which rank among the most successful in the history of mathematics. A posteriori, we see today that the parallel developments of the fifties were an essential step for the syntheses which followed.

GENTLEMAN. To return to prime numbers, we accept that there is an infinite number of them, and consequently there is an infinite number of inverses of these primes. Is it true that the sum of these inverses is finite?

SERGE LANG. That's a very nice question! You want to take the sum

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$$

GENTLEMAN. Yes.

SERGE LANG. So it's the sum

$$\sum_{p \leqq x} \frac{1}{p}.$$

Well, if I wanted to plant someone in the audience to ask a question which fit exactly with what I said before, I could not have done better than to have planted the gentleman over there. [*Laughter.*]

Remember that our product was

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right).$$

We have just written a sum with $1/p$. The two look like each other. One of them looks multiplicative, and the other looks additive. but the fact that they look alike is not at all accidental, and is due precisely to the logarithm which I did not have time to discuss much. But if you give me two minutes . . . The logarithm has two simple properties. The first is that

$$\log (ab) = \log a + \log b.$$

In other words, the logarithm of a product is equal to the sum of the logs. If you know the logarithm, you know this property.

The second property is that when t is very small, then $\log(1+t)$ is approximately equal to t. Therefore $\log(1-t)$ is approximately equal to $-t$.

Now suppose that I take the logarithm of the product. Since the log of a product is equal to the sum of the logs, we have

$$\log \prod_{p \leqq x} \left(1 - \frac{1}{p}\right) = \sum_{p \leqq x} \log \left(1 - \frac{1}{p}\right).$$

But $\log(1 - 1/p)$ is approximately equal to $-1/p$. Hence our sum is approximately equal to

$$\sum_{p \leqq x} \log \left(1 - \frac{1}{p}\right) \sim - \sum_{p \leqq x} \frac{1}{p},$$

which is precisely the sum which the gentleman wants to consider. It is a theorem which ones proves when you analyze the sum that we have the asymptotic relation

$$\sum_{p \leqq x} \frac{1}{p} \sim \log \log x.$$

Since the logarithm grows very slowly, the iterated logarithm $\log \log x$ grows even more slowly. But it grows, and the sum is very interesting. So it is not true that the sum of the inverses $1/p$, taken for all primes p, is finite.

You see, if you study that sum, you find $\log \log x$. To study the product, you perform the inverse operation, you exponentiate, and you find $\log x$, always with a minus sign. So you find that

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right) \text{ is approximately equal to } \frac{1}{\log x},$$

which is precisely what we had before. All of this belongs to the same circle of ideas. The gentleman gets an A+.

QUESTION. Do you see applications of prime number theory in the sciences?

SERGE LANG. The sciences? You mean physics, chemistry, biology? I don't know any, but the history of mathematics shows that subjects which were considered pure can, at any moment, have the most unexpected concrete applications. I cannot predict in advance what will happen. I don't know any, but that does not mean that there aren't any, because I know practically nothing about physics and chemistry. There may be applications which I don't know about. On the other hand, I can't predict that there won't be any, and in fact, I do exactly the opposite: I say that there may be some, at any time. For instance, these last few years, pure mathematical theories in differential geometry or topology which were discovered ten or twenty years ago suddenly found applications to the theory of elementary particles in physics!

I try to avoid absolutes, from one side or another. I have told you what I like, I show you what I like. And I hope that you like it. And if it works like that, it's all I wanted to do.

Addendum

QUESTION. And the Riemann hypothesis, which you mentioned before. Can you tell us what it is?

SERGE LANG. Yes. We want to give a more precise description of the error term in the formula for the number of primes. The term $x/\log x$ is only a very gross approximation, even asymptotically. There is another expression which gives a much better approximation.

Remember that we had found a certain fraction

$$e^{\gamma} F(x), \quad \text{or also } \frac{1}{\log x}$$

which we shall now call the density of primes, or also the probability that x is prime, asymptotically. After that, we said that $\pi(x)$ is asymptotic to the product of this density with x, that is

$$\pi(x) \sim \frac{1}{\log x} x.$$

But we can do better than to take this product, because $\log x$ varies with x. We get a much better formula by taking the sum of the densities, from 2 to x, which we denote by $L(x)$. That means we let

$$L(x) = \frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \frac{1}{\log 5} + \cdots + \frac{1}{\log x}$$
$$= \sum_{n=2}^{x} \frac{1}{\log n}.$$

Then we also have the asymptotic relation

$$\pi(x) \sim L(x) \sim \frac{x}{\log x},$$

but $L(x)$ gives a much better approximation of $\pi(x)$ than $x/\log x$. The Riemann Hypothesis states that

$$\pi(x) = L(x) + O(\sqrt{x} \log x),$$

where $O(\sqrt{x} \log x)$ is an error term, bounded by $C\sqrt{x} \log x$, where C is some constant. Since $\sqrt{x}$ and $\log x$ are very small compared to x, we see that $L(x)$ gives a very good approximation to $\pi(x)$.

The Riemann Hypothesis also allows us to understand better the relation between the product $F(x)$ and $1/\log x$. Indeed, H. Montgomery tells me that it implies the relation

$$e^{\gamma} F(x)x = \frac{x}{\log x} + O(\sqrt{x}),$$

where again $O(\sqrt{x})$ is an error term bounded by $C\sqrt{x}$, with some suitable constant C. Hence the expressions $e^{\gamma} F(x)x$ and $x/\log x$ give about the same approximation to $\pi(x)$, and both are worse than $L(x)$.

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