COHOMOLOGY OF LINE BUNDLES ON THE COTANGENT BUNDLE OF A GRASSMANNIAN

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To Professor Shoji, on the occasion of his 60th birthday.

ABSTRACT. We show that certain line bundles on the cotangent bundle of a Grassmannian arising from an anti-dominant character \( \lambda \) have cohomology groups isomorphic to those of a line bundle on the cotangent bundle of the dual Grassmannian arising from the dominant character \( w_0(\lambda) \), where \( w_0 \) is the longest element of the Weyl group of \( SL_{l+1}(k) \).

1. INTRODUCTION

Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Consider the algebraic group \( G = SL_{l+1}(k) \). Let \( T \subset B \) be a maximal torus contained in a Borel subgroup of \( G \) and let \( X^*(T) \) denote the characters of \( T \). We choose positive roots and simple roots \( \Pi \) in \( X^*(T) \) which correspond to the Borel subgroup opposite to \( B \). We index \( \Pi = \{ \alpha_j \} \) so that \( \alpha_1 \) is an extremal root and \( \alpha_j \) is next to \( \alpha_{j+1} \) in the Dynkin diagram of type \( A_l \). Let \( \{ \omega_i \} \) be the fundamental weights of \( G \) corresponding to \( \Pi \). Let \( \alpha^\vee \) be the coroot of the root \( \alpha \) and let \( \langle -, - \rangle \) denote the pairing of \( X^*(T) \) and the cocharacters \( X_*(T) \) of \( T \).

For a rational representation \( V \) of \( B \), denote by \( H^*(G/B, V) \), or just \( H^*(V) \) when there is no ambiguity, the cohomology of the sheaf of sections of the vector bundle \( G \times_B V \) over \( G/B \). For \( \lambda \in X^*(T) \), we use the notation \( \lambda \) both for a character of \( T \) and for the one-dimensional representation of \( T \) or \( B \) that it defines.

Let \( P \) denote the maximal proper parabolic subgroup containing \( B \) corresponding to all the simple roots except \( \alpha_m \). Thus \( G/P \) identifies with the Grassmannian of \( m \)-planes in \( l+1 \)-space. Let \( u_m \) be the Lie agebra of the unipotent radical of \( P \). Denote by \( S^nu_m^\ast \) the \( n \)-th symmetric power of the linear dual of \( u_m \).

The result of this paper is the following:

Theorem 1.1. Let \( r \) be an integer in the range

\[-|l + 1 - 2m| - 1 \leq r \leq 0.\]

If \( p = 0 \) or

\[p > \max\{r, \min\{m, l + 1 - m\}\},\]

then there is a \( G \)-module isomorphism

\[H^i(G/B, S^nu_m^\ast \otimes r\omega_m) \simeq H^i(G/B, S^{n+rm}u_{l+1-m}^\ast \otimes -r\omega_{l+1-m}) \text{ for all } i, n \geq 0.\]
That the theorem is related to the cohomology of line bundles on the cotangent bundle of a Grassmannian goes as follows. First, since all roots have the same length, the cotangent bundle of $G/P$ identifies with the vector bundle $G \times P u_m$. Second, for $\alpha \in X^*(P)$, let $L_\alpha$ denote the corresponding line bundle on $G/P$. Let $\pi : G \times P u_m \to G/P$ be the natural map. The line bundle $L_\alpha$ can be pulled back to a line bundle $\pi^*(L_\alpha)$ on $G \times P u_m$. We have

$$
H^i(G \times P u_m, \pi^*(L_\alpha)) \simeq \bigoplus_{n \geq 0} H^i(G/P, S^n u_m \otimes \lambda).
$$

Finally,

$$
H^i(G/B, V) \simeq H^i(G/P, V) \text{ for all } i \geq 0
$$

for any $P$-representation $V$. See Chapter 8 of Jantzen’s notes [3] or [1] for a discussion of these facts, which hold for any parabolic subgroup of a semisimple group.

Theorem 1.1 was used in the papers [4] and [5] to prove that certain nilpotent varieties are normal. An analogous theorem for $G$ of type $D_{2t+1}$ was proved and used in [5], although that theorem was stated only in characteristic zero. The usefulness of Theorem 1.1 lies in the fact that it can be used for arbitrary semisimple $G$ whenever $P$ is replaced by a parabolic subgroup of $G$ whose Levi subgroup $L$ contains a semisimple subgroup $M$ of type $A_{m-1} \times A_{t-m}$ such that $G$ contains a Levi subgroup $L'$ of semisimple type $A_t$ where $M \subset L'$ and $[L', L] \subset L$. Then multiple applications of Theorem 1.1 often lead to a situation, at least in characteristic zero, where the cohomology groups vanish for all $n \geq 0$ when $i > 0$. The reason is that when $p = 0$ we are in a position to invoke a version of the Grauert-Riemenschneider theorem.

Perhaps the most interesting feature of the theorem is that it translates the cohomology of a line bundle on the cotangent bundle of one partial flag variety into the cohomology of a line bundle on the cotangent bundle of a different partial flag variety.

2. Preliminaries

Recall the following proposition, due to Demazure and extended to positive characteristic by Thomesen. It is true for all semisimple groups, although we use it here only for type $A$. From now on, $P_\alpha$ refers to the minimal parabolic subgroup of $G$ containing $B$ corresponding to the simple root $\alpha$. If $\alpha = \alpha_\alpha$, then we may write $P_t$ instead of $P_\alpha$.

**Proposition 2.1.** [2] [6] Let $V$ be a rational representation of $B$ and assume that $V$ extends to a representation of the parabolic subgroup $P_\alpha$. Let $\lambda \in X^*(T)$ be such that $s = (\lambda, \alpha_\lambda) \leq -1$. If $p = 0$ or $p > -s$, then there is a $G$-module isomorphism

$$
H^i(G/B, V \otimes \lambda) \simeq H^{i-1}(G/B, V \otimes (s+1)\alpha) \text{ for all } i \geq 0.
$$

In particular, if $s = -1$ then all cohomology groups $H^i(G/B, V \otimes \lambda)$ vanish.

This leads to the following result for $G = SL_{t+1}(k)$.

**Lemma 2.2.** Let $Q$ be a representation of $B$ that extends to a representation of each $P_t$ for $a \leq t \leq b$. Let $\lambda \in X^*(T)$ be such that $\langle \lambda, \alpha_a \rangle = 0$ for $a < t \leq b$. Set $s = (\lambda, \alpha_a)$ and assume that $a - b - 1 \leq s \leq -1$. If $p = 0$ or $p > -s$, then $H^*(Q \otimes \lambda) = 0$.

**Proof.** This is an application of Proposition 2.1, utilizing it a total of $-s$ times, starting with the parabolic $P_a$. After one application we have

$$
H^i(Q \otimes \lambda) \simeq H^{i-1}(Q \otimes (s-1)\alpha_a).
$$

After the second application we have that the latter is isomorphic to

$$
H^{i-2}(Q \otimes (s-1)\alpha_a + (-s-2)\alpha_{a+1}).
$$

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1The reference for the theorem in those papers is supplanted by the current paper.
Continuing along, we find that after $-s - 1$ times that

$$H^i(Q \otimes \lambda) \simeq H^{i+s+1}(Q \otimes \lambda + (-s - 1)\alpha_a + \cdots + 2\alpha_{a-s-3} + \alpha_{a-s-2}).$$

This is possible since $a - s - 2 < b$ and thus $Q$ extends to a representation of $P_j$ for $a \leq j \leq a - s - 2 < b$.

We are done at this point because

$$\langle \lambda + (-s - 1)\alpha_a + \cdots + \alpha_{a-s-2}, \alpha_{a-s-1}^\vee \rangle = -1$$

and $a - s - 1 \leq b$ so that $Q$ is a representation of $P_{a-s-1}$. Thus Proposition 2.1 applies again, giving the total vanishing of cohomology. 

We can now prove the main result.

3. PROOF OF THEOREM 1.1

Proof. By symmetry we may assume that $m \leq l + 1 - m$, so that the extremal value for $r$ is

$$-|l + 1 - 2m| - 1 = 2m - l - 2.$$

Step 1. In this step, $r$ may be an arbitrary integer. Consider the intersection $V = u_m \cap u_{l+1-m}$. We will show in Step 1 that for all $i, n$

$$H^i(S^n u_m^* \otimes r\omega_m) \simeq H^i(S^n V^* \otimes r\omega_m).$$

We begin by taking the Koszul resolution of the short exact sequence

$$0 \to U \to u_m^* \to V^* \to 0$$

(this defines $U$) and tensoring it with $r\omega_m$. This gives

$$0 \to \cdots \to S^n u_m^* \otimes \wedge^j U \otimes r\omega_m \to \cdots \to S^n u_m^* \otimes r\omega_m \to S^n V^* \otimes r\omega_m \to 0.$$

We claim that

$$H^*(S^n u_m^* \otimes \wedge^j U \otimes r\omega_m) = 0$$

for $1 \leq j \leq \dim U$ from which Equation 1 will follow. The $T$-weights of $U$ are those of the form

$$\alpha_c + \alpha_{c+1} + \cdots + \alpha_d,$$

where $c \leq m$ and $m \leq d < l + 1 - m$. Therefore, if $\lambda$ is a $T$-weight of $\wedge^j U$, there exists $a$ with $m < a \leq l + 1 - m$ such that $-m \leq \langle \lambda, \alpha_a^\vee \rangle \leq -1$ and $\langle \lambda, \alpha_i^\vee \rangle = 0$ for $t > a$.

Set $s = \langle \lambda, \alpha_a^\vee \rangle$ and $b = 1$. We can invoke Lemma 2.2 for $\lambda$ and $Q := S^n u_m^* \otimes r\omega_m$. Indeed, $Q$ is stable under the parabolic subgroups $P_t$ for $t \geq a$. Also $a - b - 1 \leq -m$ since $a \leq l + 1 - m$ and so $s$ is in the range $a - b - 1 \leq -m \leq s \leq -1$ and so the lemma applies, given our assumption on the characteristic of $k$. It follows that $H^*(Q \otimes \lambda) = 0$ for all weights $\lambda$ appearing in $\wedge^j U$ for $1 \leq j \leq \dim U$. Thus if we filter $\wedge^j U$ by $B$-subrepresentations such that the consecutive quotients are one-dimensional, we deduce that $H^*(Q \otimes \wedge^j U) = 0$ for $1 \leq j \leq \dim U$ and Equation 1 follows.

Step 2.

Let $V_1 = V \cap u_{m-1}$ and $V_2 = V \cap u_{l+2-m}$. If $m = 1$, then $u_{m-1}$ and $u_{l+2-m}$ are considered to be the zero vector space. Let $\mu$ be a weight of the form

$$r\omega_m + r'\omega_{l+1-m}$$

and assume that $2m - 2 - l \leq r \leq -1$ with $r'$ unrestricted, unless $r = 2m - 2 - l$, in which case assume that $r' = 0$. In this step we show for all $n \geq 0$ that

$$H^*(S^n V_1^* \otimes \mu) = 0$$

Take the Koszul resolution of

$$0 \to U_2 \to V_1^* \to V_2^* \to 0.$$
(this defines $U_2$) and tensor it with $\mu$. We will show that

$$H^*(S^nV_2^* \otimes \mu) = 0$$

and

$$H^*(S^{n-j}V_1^* \otimes \land^jU_2 \otimes \mu) = 0$$

for $1 \leq j \leq m-1$ and then Equation 2 will follow (the dimension of $U_2$ is $m-1$ as shown below).

The subspace $V_2$ coincides with $u_{m-1} \cap u_{l+2-m}$. Hence $V_2^*$ is stable under $P_t$ for $m \leq t \leq l+1-\ell$. It follows that $H^*(S^nV_2^* \otimes \mu) = 0$ by Lemma 2.2 with $a = m$, $b = l - m$ unless $r' = 0$ in which case $b = l + 1 - m$. In all cases, we have $a - b - 1 \leq r \leq -1$ by hypothesis and the lemma applies since we are assuming $p > -r$.

Now the weights of $U_2$ are

$$\alpha_c + \alpha_{c+1} + \cdots + \alpha_{l+1-m}$$

where $1 \leq c \leq m - 1$. If $\lambda$ is a weight of $\land^jU_2$, then $\lambda$ satisfies $\langle \lambda, \alpha_{l+2-m} \rangle = -j$ and $\langle \lambda, \alpha_1 \rangle = 0$ for $t > l + 2 - \ell$. Consequently, if we filter $\land^jU_2$ as in Step 1 and apply Lemma 2.2, we get

$$H^*(S^{n-j}V_1^* \otimes \land^jU_2 \otimes \mu) = 0$$

for $1 \leq j \leq m-1$. The lemma works with $a = l + 2 - m$, $b = l$. Thus $a - b - 1 = (l+2-m) - l - 1 = -m+1$ and $j$ is in the acceptable range $-m + 1 \leq -j \leq -1$. We are also using the fact that $S^{n-j}V_1^* \otimes \mu$ is stable under $P_t$ for $t \geq l + 2 - m$.

**Step 3.** In this step, we show that for all $i, n$

$$(3) \quad H^i(S^nV^* \otimes \mu) \simeq H^i(S^{n-m}V^* \otimes \mu + \omega_m + \omega_{l+1-m})$$

for $\mu$ as in Step 2.

We take the Koszul resolution of the short exact sequence

$$0 \to U_1 \to V^* \to V_1^* \to 0$$

(this defines $U_1$) and tensor it with $\mu$ to get

$$(4) \quad 0 \to S^{n-m}V^* \otimes \land^mU_1 \otimes \mu \to \cdots \to S^{n-j}V^* \otimes \land^jU_1 \otimes \mu \to \cdots \to S^nV^* \otimes \mu \to S^nV_1^* \otimes \mu = 0$$

The weights of $U_1$ are of the form

$$\alpha_m + \alpha_{m+1} + \cdots + \alpha_d$$

where $l + 1 - m \leq d \leq \ell$ (and in particular, $\dim U_1 = m$). We study the terms $\land^jU_1$ for $1 \leq j < m$. If $\lambda$ is a weight of $\land^jU_1$, then $\lambda$ satisfies $\langle \lambda, \alpha_{m-j} \rangle = -j$ and $\langle \lambda, \alpha_{l} \rangle = 0$ for $t < m - 1$. Consequently, proceeding as in Step 1, we filter $\land^jU_1$ and apply Lemma 2.2 (after applying an outer automorphism to $G$ to arrive at the obvious symmetric set-up) to get

$$H^*(S^{n-j}V^* \otimes \land^jU_1 \otimes \mu) = 0$$

when $j < m$. The lemma works with $a = 1$, $b = m - 1$, so that $a - b - 1 = -m + 1 \leq -j \leq -1$. We note that $V$ is a representation of $P_t$ for $t \leq m - 1$.

On the other hand, for the case $j = m$, we have

$$\land^mU_1 = m(\alpha_m + \alpha_{m+1} + \cdots + \alpha_{l+1-m}) + (m-1)\alpha_{l+2-m} + \cdots + 2\alpha_{l-1} + \alpha_l.$$

So $m - 1$ applications of Proposition Demazure as in the proof of Lemma 2.2 yields

$$H^i(S^{n-m}V^* \otimes \land^mU_1 \otimes \mu) \simeq H^{i+m-1}(S^{n-m}V^* \otimes \mu + \omega_m + \omega_{l+1-m}).$$

By breaking Equation 4 into short exact sequences and taking their cohomology, we conclude that

$$H^i(S^nV^* \otimes \mu) \simeq H^i(S^{n-m}V^* \otimes \mu + \omega_m + \omega_{l+1-m}),$$

where we are using

$$H^*(S^nV_1^* \otimes \mu) = 0$$
from Step 2.

**Step 4.** We obtain the theorem by using Step 3 repeatedly, starting with $\mu = r\omega_m$ with $r$ in the prescribed range of the statement of the theorem. After $-r$ steps we arrive at

$$H^i(S^nV^* \otimes r\omega_m) \cong H^i(S^{n+rm}V^* \otimes -r\omega_{l+1-m}),$$

for all $i,n$. The proof is completed by using Step 1 and the symmetric version of Equation 1 which gives

$$H^i(S^{n+rm}V^* \otimes -r\omega_{l+1-m}) \cong H^i(S^{n+rm}u^*_{l+1-m} \otimes -r\omega_{l+1-m})$$

for all $i,n$. 

\[\square\]

**References**


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