A FAMILY OF AFFINE WEYL GROUP REPRESENTATIONS

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ABSTRACT. In this paper we explicitly determine the virtual representations of the finite Weyl subgroups of the affine Weyl group on the cohomology of the space of affine flags containing a family of elements n_t in an affine Lie algebra. We also compute the Euler characteristic of the space of partial flags containing n_t and give a connection with hyperplane arrangements.

1. INTRODUCTION

Let W_a be an affine Weyl group associated to a Weyl group W of rank r. For $w \in W_a$ let $s(w)$ be the least number of reflections needed to write w as a product of reflections. In the first part of the paper, we construct a permutation representation U_t of W_a of dimension t^r for t a natural number. This representation has the property that for w of finite order, the trace of w on U_t equals $t^{r-s(w)}$ when t is not divisible by certain primes.

We then recall the construction due to Lusztig of certain regular semisimple nil-elliptic elements n_t in an affine Lie algebra. Fan has shown that the Euler characteristic of the space of affine flags containing n_t is t^r, extending a result of Lusztig–Smelt in type A_r, see [F], [LS]. We generalize this to the space of partial affine flags containing n_t. Our result is that the Euler characteristic is

$$\frac{(t+m_1)(t+m_2)...(t+m_j)t^{r-j}}{|W^J|}$$

where W^J is the finite Weyl subgroup of W_a corresponding to the partial flag manifold $\hat{\mathcal{B}}^J$ and $m_1, \ldots, m_j$ are the exponents of W^J. The referee has pointed out that D. S. Sage proved this result for the classical groups [Sa].

Let U_t also denote the restriction of U_t to W^J. We show that U_t decomposes into a direct sum of representations induced from a parabolic subgroup of W^J. This yields a connection with the characteristic polynomials of certain hyperplane arrangements [OS2]. Our construction gives a new way to determine these characteristic polynomials, which were determined by Orlik and Solomon in [OS2].

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Finally using results of Lusztig and Alvis–Lusztig [AL], we show that the virtual representation of W^J on the cohomology of the space of affine flags containing n_t is U_t.

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2. Notation

Let Φ be an irreducible root system of rank r which spans the Euclidean space V. The inner product $(\ ,\ )$ on V is assumed to be invariant under the Weyl group W of Φ. The coroots $\Phi^\vee$ are those elements of V of the form $2\alpha/(\alpha,\alpha)$ where $\alpha \in \Phi$. Let $L = L(\Phi^\vee)$ be the lattice in V generated by $\Phi^\vee$ and let

$$\hat{L} = \hat{L}(\Phi^\vee) = \{v \in V \,|\, (v,\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}.$$

We define $f = [\hat{L} : L]$, the index of L in $\hat{L}$.

Let $\Pi = \{\alpha_i\} \subset \Phi^+$ be a set of simple roots contained in a set of positive roots and let θ be the highest root in Φ^+. Set $\alpha_0 = -\theta$ and let $\tilde{\Pi} = \Pi \cup \{\alpha_0\}$. Define the coefficients c_i of θ by the equation $\theta = \sum_{i=1}^r c_i\alpha_i$ and set $c_0 = 1$.

Let L act on V by translation and form the affine Weyl group $W_a = W \ltimes L$. Let $H_{\alpha,k} = \{v \in V \,|\, (v,\alpha) = k\}$ where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Then it is known [H] that W_a is generated by the reflections $s_{\alpha,k}$ in the hyperplanes $H_{\alpha,k}$. Let $s(w)$ be the least number of reflections needed to write $w \in W_a$ as a product of reflections. If w is of finite order, then w has a fixed point on V. In this case define $d(w)$ to be the dimension of the fixed point set of w. Then $d(w) = r - s(w)$, see [C].

We study the natural action of W_a on the set $S_t = L/tL$ where $t \in \mathbb{N}$. Thus we get a representation U_t of W_a of dimension t^r on the space of complex functions on S_t. Similarly, we can consider the action of W_a on the set $\hat{S}_t = \hat{L}/tL$ and get a representation $\hat{U}_t$ of dimension ft^r.

3. Fixed points of W_a on S_t and $\hat{S}_t$

For $w \in W_a$ of finite order, we want to know the number of fixed points of w on S_t and $\hat{S}_t$. When t is not divisible by certain primes associated to Φ, the answer is readily computed. But first we need some preliminaries.

Recall that a root subsystem of Φ is a subset of Φ which is itself a root system. Let $M \subset \Phi$. The root subsystem spanned by M is the collection of roots in Φ which are integral combinations of elements in M. The rational closure of a root subsystem Φ' denoted by $\bar{\Phi}'$ is the collection of roots in Φ which are rational combinations of elements in Φ'. Note that $\bar{\Phi}'$ is a root system of the same rank as Φ'. We will always assume our root subsystems are integrally closed, i.e., the root subsystem Φ' equals the root subsystem in Φ that it spans.

Definition 3.1. A bad prime of Φ is a prime which divides the order of the torsion subgroup of $L(\Phi)/L(\Phi')$ for some root subsystem $\Phi' \subset \Phi$.

The following two results can be found in [St].

Theorem 3.2. *The bad primes of Φ are precisely those primes which divide a coefficient of θ.*

Lemma 3.3. *Let Φ' be a root subsystem. Any set of simple roots for $\bar{\Phi}'$ can be extended to a set of simple roots for Φ.*

Corollary 3.4. *Let Φ' be a root subsystem. Then $L(\bar{\Phi}')/L(\Phi')$ is isomorphic to the torsion subgroup of $L(\Phi)/L(\Phi')$.*

Proof. Lemma 3.3 implies that $L(\Phi)/L(\bar{\Phi}')$ is torsion free. Now the corollary follows from the fact that $L(\bar{\Phi}')$ and $L(\Phi')$ have the same rank. □ □

Definition 3.5. We say t is good (for Φ) if it is prime to every bad prime. We say t is very good if it is also prime to f.

Remark 3.6. By inspection when t is prime to the Coxeter number h of W, it is very good.

Lemma 3.7. *Let $\beta_1, \dots, \beta_r$ be a set of linearly independent roots and let Φ' be the subsystem they span. Let $k_1, \dots, k_r \in \mathbb{Z}$. If t is good, there exists $u \in \hat{L}$ with $(u, \beta_i) \equiv k_i (\mathrm{mod}\ t)$ for all i. If t is very good, there exists $u \in L$ with $(u, \beta_i) \equiv k_i (\mathrm{mod}\ t)$ for all i.*

Proof. The inner product $(\ ,\ )$ induces a pairing of $\hat{L}$ and $L(\Phi')$ which has determinant equal to $[L(\Phi) : L(\Phi')]$. The pairing of L and $L(\Phi')$ has determinant equal to $f[L(\Phi) : L(\Phi')]$. The lemma follows since these determinants are invertible modulo t under the respective hypotheses. □ □

Definition 3.8. Let $\Phi' \subset \Phi$ be a root subsystem. Define

$$P(\Phi') = \{u \in S_t |\ (u, \alpha) \equiv 0 \pmod{t} \text{ for all } \alpha \in \Phi'\}$$

and similarly

$$\hat{P}(\Phi') = \{u \in \hat{L}/t\hat{L} |\ (u, \alpha) \equiv 0 \pmod{t} \text{ for all } \alpha \in \Phi'\}.$$

Assume $w \in W_a$ is of finite order. We can now prove the main result on the number of fixed points of w on S_t and $\hat{S}_t$.

Proposition 3.9. *If t is good, then the number of fixed points of w on $\hat{S}_t$ is $ft^{d(w)}$. If t is very good, then the number of fixed points of w on S_t is $t^{d(w)}$.*

Proof. We give the proof for S_t, the case of $\hat{S}_t$ being similar.

Let $l = s(w)$ and let $w = s_{\beta_1,k_1} s_{\beta_2,k_2} ... s_{\beta_l,k_l}$ be a minimal expression for w as a product of reflections. The roots β_i are necessarily linearly independent. Let Φ' be the root subsystem of rank l that they span. For any $u \in L$, continue to denote by u its image in S_t. By an easy induction on $s(w)$, we have

$$w(u) = u \text{ if and only if } (u, \beta_j) \equiv k_j \pmod{t} \text{ for } j = 1, 2, ..., l.$$

Applying the previous lemma, we conclude that the number of fixed points of w is just $|P(\Phi')|$.

Because $f = [\hat{L} : L]$ and t is prime to f, the inclusion of L into $\hat{L}$ induces an isomorphism of L/tL and $\hat{L}/t\hat{L}$. This isomorphism maps $P(\Phi')$ onto $\hat{P}(\Phi')$.

Now t is not divisible by any bad prime, so Corollary 3.4 implies t is prime to $[L(\bar{\Phi}') : L(\Phi')]$. It is easy to see that in this case $\hat{P}(\Phi') = \hat{P}(\bar{\Phi}')$.

We are reduced to computing the cardinality of $\hat{P}(\bar{\Phi}')$. Extend a set of simple roots $\{\lambda_1, \dots, \lambda_l\}$ of $\bar{\Phi}'$ to a set of simple roots $\{\lambda_1, \dots, \lambda_l\} \cup \{\lambda_{l+1}, \dots, \lambda_r\}$ of Φ as in Lemma 3.3. Let $\omega_1^{\vee}, \omega_2^{\vee}, \dots, \omega_r^{\vee} \in \hat{L}$ be a corresponding set of fundamental coweights. That is, $(\omega_i^{\vee}, \lambda_j) = \delta_{ij}$. Then we have

$$\hat{P}(\bar{\Phi}') = \{\sum_{i=1}^{r} x_i \omega_i^{\vee} \in \hat{L}/t\hat{L} |\, x_j \equiv 0 \pmod{t} \text{ for } j = 1, 2, \dots, l\}.$$

Clearly, the cardinality of $\hat{P}(\bar{\Phi}')$ is just $t^{r-l} = t^{d(w)}$. □ □

Remark 3.10. The results in this section can be extended to root systems which are not irreducible. We will need this in the next section.

4. Stabilizers of elements in $\hat{S}_t$

Let $s_i \in W_a$ be the reflection in the hyperplane $H_{\alpha_i,0}$ for $i = 1, \dots, r$ and let $s_0 \in W_a$ be the reflection in the hyperplane $H_{\theta,1}$. Let $I = \{s_0, s_1, \dots, s_r\}$. For any proper subset J of I, the subgroup of W_a generated by the elements in J is a finite Weyl group W^J corresponding to a (not necessarily irreducible) root system Φ^J. Recall that a parabolic subgroup of W^J is a subgroup of W^J that is W^J-conjugate to $W^{J'}$ for some $J' \subset J$.

In this section we prove the following proposition.

Proposition 4.1. *If t is good, the stabilizer in W^J of an element in $\hat{S}_t$ (or S_t) is a parabolic subgroup of W^J.*

First, we need some lemmas. Let $Q_1 = W^J, Q_2, \dots, Q_k$ be representatives of the conjugacy classes of subgroups of W^J which are Weyl groups of root subsystems of Φ^J.

Lemma 4.2. *Let U be a representation of W^J. Suppose U has two expressions as a sum of induced representations*

$$U = \oplus_{i=1}^{k} f_i \mathrm{Ind}_{Q_i}^{W^J}(1) = \oplus_{i=1}^{k} f_i' \mathrm{Ind}_{Q_i}^{W^J}(1)$$

where f_i and f_i' are nonnegative integers and $f_i = 0$ whenever Q_i is not a parabolic subgroup of W^J. Then $f_i = f_i'$ for $i = 1, 2, \dots, k$.

Proof. Let w_j be a Coxeter element of Q_j. In general it is possible for w_j to be conjugate to $w_{j'}$ when $j \neq j'$. This occurs in B_4, for example. Nevertheless the elements w_j distinguish the subgroups Q_j of W^J enough to arrive at the conclusion of the lemma.

In fact the following statements are true about w_j:

1. $\mathrm{tr}(w_j, \mathrm{Ind}_{Q_i}^{W^J}(1)) \geq 0$ and $\mathrm{tr}(w_j, \mathrm{Ind}_{Q_j}^{W^J}(1)) > 0$.
2. $\mathrm{tr}(w_j, \mathrm{Ind}_{Q_i}^{W^J}(1)) = 0$ if $\mathrm{rank}Q_i < \mathrm{rank}Q_j$.
3. If Q_j is a parabolic subgroup of W^J, then $\mathrm{tr}(w_j, \mathrm{Ind}_{Q_i}^{W^J}(1)) = 0$ if $\mathrm{rank}Q_i \leq \mathrm{rank}Q_j$ and $j \neq i$.

Only the last statement is not obvious. It is equivalent to the statement that no conjugate of w_j belongs to any Q_i of the same rank as Q_j for $i \neq j$. This statement was checked by comparing the characteristic polynomial for w_j with the characteristic polynomial of elements in Q_i using the analysis of conjugacy classes in a Weyl group given in [C].

The lemma follows from these statements by reverse induction on the rank of the subgroups Q_i by taking the trace of w_j on both expressions for U. □ □

Lemma 4.3. *Let M be a proper subset of $\tilde{\Pi}$. Let Φ' be the root subsystem spanned by M. Then M is a set of simple roots for Φ'.*

Proof. For any $\alpha \in V$, write $\alpha \succ 0$ (resp. $\prec 0$) if α is a linear combination of elements of Π with all positive (resp. all negative) coefficients. Let $\alpha \in \Phi'$ and assume $\alpha \in \Phi^+$. We can write $\alpha = \sum_{\beta \in M} d_\beta \beta$ where $d_\beta \in \mathbb{Z}$. We need to show that all $d_\beta \geq 0$ or all $d_\beta \leq 0$. Rewriting, we have

$$\alpha + d_{-\theta}\theta = \sum_{\beta \in M \backslash \{-\theta\}} d_\beta \beta.$$

Assume $d_{-\theta} \geq 0$. Then $\alpha + d_{-\theta}\,\theta \succ 0$ which forces all $d_\beta \geq 0$ because $M \backslash \{-\theta\} \subset \Pi$. So assume $d_{-\theta} < 0$. Because θ is the highest root of Φ, we have $\alpha + d_{-\theta}\,\theta \prec 0$. This forces all $d_\beta \leq 0$. Repeat for $-\alpha$ if $\alpha \notin \Phi^+$.
□ □

Corollary 4.4. *Let M be a proper subset of $\tilde{\Pi}$. Let Φ' be the root subsystem of Φ spanned by M. Then M can be extended to a set of simple roots for Φ if and only if $\Phi' = \bar{\Phi}'$.*

Proof. Assume $\Phi' = \bar{\Phi}'$. By the previous lemma M is a set of simple roots for $\Phi' = \bar{\Phi}'$ and by Lemma 3.3 any set of simple roots for $\bar{\Phi}'$ can be extended to a set of simple roots for Φ. It is immediate that the converse is true.
□ □

Remark 4.5. The corollary makes it easy to see when a proper subset M of $\tilde{\Pi}$ extends to a set of simple roots for Φ. This is the case if and only if the coefficients of θ on the roots *not* in M are relatively prime.

We begin by proving the proposition for the case $J = \{s_1, \ldots, s_r\}$, so that $W^J = W$. First, we need some more definitions.

Denote by W_t the subgroup of W_a of the form $W \ltimes tL$. Note that W_t is isomorphic to $W_a = W_1$ for all t. Let

$$D_t = \{u \in V |\, (u, \alpha) \geq 0 \text{ for } \alpha \in \Pi \text{ and } (u, \theta) \leq t\}.$$

Recall that D_1 is a fundamental domain for the action of W_a on V (see [H]). The same proof also shows that D_t is a fundamental domain for the action of W_t on V for any t. Let $\hat{L}_t = D_t \cap \hat{L}$. Then each W-orbit on $\hat{S}_t$ contains a unique element of $\hat{L}_t$.

Proof of Proposition 4.1 *when* $W^J = W$. Choose $u \in \hat{S}_t$ and let W_u be the stabilizer of u in W. Without loss of generality, we can assume that $u \in \hat{L}_t$. Let $\Pi_u = \{\alpha \in \tilde{\Pi} | \, (u, \alpha) \equiv 0 \, (\text{mod } t)\}$ and let Φ' be the root subsystem spanned by Π_u. It is clear that W_u is just the Weyl group of Φ'. To show that W_u is a parabolic subgroup of W, we must show that a set of simple roots for Φ' extends to a set of simple roots for Φ. Hence by Corollary 4.4, we must show that $\Phi' = \bar{\Phi}'$.

We may assume that u is nonzero and $\Pi_u \not\subset \Pi$, the result being clear otherwise. So $-\theta \in \Pi_u$. Since $u \in \hat{L}_t$ is nonzero and $-\theta \in \Pi_u$, we must have $(u, \theta) = t$. Also either $(u, \alpha) = 0$ or t for each $\alpha \in \Pi_u \backslash \{-\theta\}$. But if $(u, \alpha') = t$ for some $\alpha' \in \Pi_u \backslash \{-\theta\}$, then $(u, \theta) = t$ forces $(u, \alpha) = 0$ for all $\alpha \in \Pi \backslash \{\alpha'\}$. In other words, $\Pi_u = \tilde{\Pi}$ and $W_u = W$. Note that this situation can only occur when the coefficient of θ on α' is one (and this can only occur when $f > 1$). We are thus reduced to the case where $(u, \alpha) = 0$ for each $\alpha \in \Pi_u \backslash \{-\theta\}$.

Now take $\beta \in \bar{\Phi}'$. Then

$$\beta = d(-\theta) + \sum_{\alpha \in \Pi_u \backslash \{-\theta\}} d_\alpha \alpha$$

where $d_\alpha, d \in \mathbb{Q}$. The fact that t is prime to the index of $L(\Phi')$ in $L(\bar{\Phi}')$ implies that (u, β) is a multiple of t. Taking the inner product with u on both sides of the expression for β reveals that d is actually an integer. Now it follows that all d_α are also integers. This means that β belongs to Φ' which is what we wanted. $\square$

Remark 4.6. The above proof extends to root systems that are not irreducible, a fact we will now use.

Proof of Proposition 4.1 *for general* W^J. Let $\hat{V}_t$ be the representation we called $\hat{U}_t$ in the case where Φ is replaced by Φ^J. Continue to denote by $\hat{V}_t, \hat{U}_t$ the restrictions of these representations to W^J. We note that if t is good for the root system Φ, then t is also good for the root system Φ^J, see [St]. Hence by the character formula of Proposition 3.9 (which only depended on the function $s(w)$ and the rank of W^J), we know that $\hat{U}_t$ is isomorphic to the direct sum of $t^{r-|J|}$ copies of $\hat{V}_t$.

Because both $\hat{U}_t$ and $\hat{V}_t$ were constructed as permutation representations (in different ways), we can express them as a direct sum of induced representations. Each W^J-orbit on the set used to define the permutation representation contributes a term $\text{Ind}_H^{W^J}(1)$ where H is the stabilizer of a point in the orbit.

Thus on the one hand, knowing the proposition for the case $W^J = W$, allows us to conclude that $\hat{V}_t = \oplus_{i=1}^k f_i \operatorname{Ind}_{Q_i}^{W^J}(1)$ where $f_i = 0$ when Q_i is not a parabolic subgroup of W^J. On the other hand it is clear that the stabilizer in W^J of an element $u \in \hat{S}_t$ is the Weyl group of a root subsystem of Φ^J. Hence we can write $\hat{U}_t = \oplus_{i=1}^k f_i' \operatorname{Ind}_{Q_i}^{W^J}(1)$. By Lemma 4.2, we can conclude that $f_i' = 0$ when Q_i is not a parabolic subgroup of W^J. In other words, the stabilizer in W^J of $u \in \hat{S}_t$ is actually a parabolic subgroup, which concludes the proof. □

For the remainder of this section and the next section, we focus on the case $W^J = W$. Let $P_1 = W, P_2, \dots, P_m$ be representatives of the conjugacy classes of parabolic subgroups of W, with $|P_i| \geq |P_j|$ for $i < j$. By Proposition 4.1 we have

$$\begin{aligned} U_t &= \oplus_{i=1}^m \chi_i(t) \operatorname{Ind}_{P_i}^W(1), \\ \hat{U}_t &= \oplus_{i=1}^m \hat{\chi}_i(t) \operatorname{Ind}_{P_i}^W(1). \end{aligned} \tag{1}$$

The functions $\chi_i(t)$ and $\hat{\chi}_i(t)$ are both well defined by Lemma 4.2. When t is very good, we have $\hat{\chi}_i(t) = f\chi_i(t)$ by comparison of the characters of U_t and $\hat{U}_t$. We will see in the next section that $\hat{\chi}_i(t)$ is a polynomial in t when t is good by relating it to the characteristic polynomial of a hyperplane arrangement. For now let us observe that $\hat{\chi}_i(t)$ has a nice combinatorial description when t is good.

Let M be a subset of $\tilde{\Pi}$. Define $p(M,t)$ to be the number of solutions $\mathbf{y}$ in strictly positive integers to the equation

$$\sum_{\alpha_i \in \tilde{\Pi} - M} c_i y_i = t.$$

Proposition 4.7. *Assume t is good. Let Π_j be a set of simple roots corresponding to P_j. Then $\hat{\chi}_j(t)$ is equal to*

$$\sum p(M,t)$$

where the sum is over the subsets M of $\tilde{\Pi}$ which are W-conjugate to Π_j.

Proof. Recall the definitions of W_u and Π_u in the first part of the proof of the previous proposition. We have

$$\begin{aligned} \hat{\chi}_j(t) &= \#\{u \in \hat{L}_t \mid W_u \text{ conjugate to } P_j\} \\ &= \#\{u \in \hat{L}_t \mid \Pi_u \text{ conjugate to } \Pi_j\} \\ &= \sum \#\{u \in \hat{L}_t \mid \Pi_u = M\} \end{aligned}$$

where the sum is over the subsets M of $\tilde{\Pi}$ which are W-conjugate to Π_j.

But $\#\{u \in \hat{L}_t \mid \Pi_u = M\}$ is easily determined. Let $\omega_1^\vee, \omega_2^\vee, \dots, \omega_r^\vee$ be a set of fundamental coweights for $\hat{L}$ corresponding to Π. Express $u \in \hat{L}$ as $y_1\omega_1^\vee + y_2\omega_2^\vee + \cdots + y_r\omega_r^\vee$. Let $y_0 = t - \sum_{i=1}^r c_i y_i$. Then $u \in \hat{L}_t$ if and only if $y_i \geq 0$ for $i = 0, 1, \dots, r$. In order for Π_u to equal M we must have $y_i = 0$

for $\alpha_i \in M$ and $y_i > 0$ for $\alpha_i \notin M$. Hence $\#\{u \in \hat{L}_t | \, \Pi_u = M\}$ is $p(M,t)$.
□ □

5. Hyperplane arrangements

Let $\mathcal{A}$ be a set of hyperplanes in $V = \mathbf{R}^n$ such that $\cap_{H\in\mathcal{A}} H = 0$. Let $\mathcal{L} = \mathcal{L}(\mathcal{A})$ be the set of intersections of these hyperplanes. We consider $V \in \mathcal{L}$. Partially order $\mathcal{L}$ by reverse inclusion and define a Möbius function μ of $\mathcal{L}$ as follows: $\mu(X,X) = 1$ and $\sum_{X\leq Z\leq Y} \mu(Z,Y) = 0$ if $X < Y$ and $\mu(X,Y) = 0$ otherwise. The characteristic polynomial of $\mathcal{L}$ is

$$\chi(\mathcal{L},t) = \sum_{X\in\mathcal{L}} \mu(V,X) t^{\dim X}.$$

Let $\mathcal{M}$ be the complex manifold obtained by removing from $\mathbf{C}^n$ the complexification of the hyperplanes in $\mathcal{A}$. Orlik and Solomon have shown that the Poincaré polynomial

$$P(\mathcal{M},t) = \sum_{p\geq 0} \dim H^p(\mathcal{M},\mathbf{C}) \; t^p$$

is equal to $(-t)^n \chi(\mathcal{L}, -t^{-1})$. For these results see [OS1].

In our case $\mathcal{A}$ is the set of hyperplanes $H_{\alpha,0}$ where $\alpha \in \Phi$. For any $X \in \mathcal{L}$ let $\mathcal{A}^X = \{X \cap H \mid H \in \mathcal{A} \text{ and } X \not\subset H\}$. Let $\mathcal{L}^X = \mathcal{L}(\mathcal{A}^X)$ be the corresponding partially ordered set and $\chi(\mathcal{L}^X,t)$ its characteristic polynomial. In [OS2] it is proved that

$$t^{\dim X} = \sum_{Y\in\mathcal{L}, Y\geq X} \chi(\mathcal{L}^Y,t). \tag{2}$$

For each $X \in \mathcal{L}$ let P_X be the elements of W which fix X pointwise. It is known that P_X is a parabolic subgroup of W. Clearly, if P_X and P_Y are conjugate, then $\mathcal{L}^X$ and $\mathcal{L}^Y$ are isomorphic and have the same characteristic polynomial.

The next proposition relates the functions $\hat{\chi}_j(t)$ to the characteristic polynomials of hyperplane arrangements. Let X_i be the fixed point set of P_i.

Proposition 5.1. *Assume t is good. Then*

$$\hat{\chi}_j(t) = \frac{f}{[N(P_j) : P_j]} \; \chi(\mathcal{L}^{X_j},t)$$

where $N(P_j)$ is the normalizer of P_j in W.

Proof. Let w_j be a Coxeter element of P_j. Taking the trace of w_j on both sides of (1) yields

$$f t^{\dim X_j} = \sum_{i=1}^{m} \hat{\chi}_i(t) \mathrm{tr}(w_j, \mathrm{Ind}_{P_i}^W(1)). \tag{3}$$

Moreover,

$$
\begin{aligned}
\operatorname{tr}(w_j, \operatorname{Ind}_{P_i}^W(1)) &= \#\{gP_i \mid g^{-1}w_jg \in P_i\} \\
&= \#\{gP_i \mid g^{-1}P_jg \subset P_i\} \\
&= \#\{gP_i \mid P_j \subset gP_ig^{-1}\} \\
&= \#\{\text{conjugates of } P_i \text{ containing } P_j\}[N(P_i) : P_i].
\end{aligned} \tag{4}
$$

On the other hand we can write equation (2) in terms of parabolic subgroups which yields

$$
\begin{aligned}
t^{\dim X_j} &= \textstyle\sum_{i=1}^m \chi(\mathcal{L}^{X_i}, t)\#\{Y \in \mathcal{L} \mid Y \geq X_j \text{ and } P_Y \text{ conjugate to } P_i\} \\
&= \textstyle\sum_{i=1}^m \chi(\mathcal{L}^{X_i}, t)\#\{\text{conjugates of } P_i \text{ containing } P_j\}.
\end{aligned} \tag{5}
$$

Now putting (3) and (4) together, comparing with (5), and arguing by induction on j gives

$$\hat{\chi}_j(t) = \frac{f}{[N(P_j) : P_j]}\, \chi(\mathcal{L}^{X_j}, t). \quad \square$$

$\square$

In [OS2], Orlik and Solomon computed the roots of $\chi(\mathcal{L}^{X_j}, t)$, which turn out to be positive integers. When $X_j = V$, the roots are the exponents of W. Propositions 5.1 and 4.7 give a different more elementary way to compute $\chi(\mathcal{L}^{X_j}, t)$. We illustrate this for the classical root systems.

Recall that Π_j is a set of simple roots corresponding to P_j and X_j is the fixed point set of P_j. A useful tool for finding the roots of $\chi(\mathcal{L}^{X_j}, t)$ is the following observation.

Let m be good and assume m is less than $\sum_{\alpha_i \in \tilde{\Pi} - M} c_i$ for all subsets $M \subset \tilde{\Pi}$ conjugate to Π_j. It follows that $p(M, m) = 0$ for all $M \subset \tilde{\Pi}$ conjugate to Π_j. Hence $\hat{\chi}_j(m) = 0$ and so m is a root of the polynomial $\chi(\mathcal{L}^{X_j}, t)$. Incidentally, this observation can be taken as a generalization of the well known fact that when m is prime to and less than h (which is just $\sum_{i=0}^r c_i$), then m is an exponent for W.

Let r_j be the cardinality of Π_j.

Proposition 5.2. *The roots of* $\chi(\mathcal{L}^{X_j}, t)$ *are* $\{1, 2, \ldots, r - r_j\}$ *for* A_r *and* $\{1, 3, \ldots, 2(r - r_j) - 1\}$ *for* B_r.

In D_r *the roots are* $\{1, 3, \ldots, 2(r - r_j) - 1\}$ *if* Π_j *is not* W*-conjugate to a set of simple roots in* $A_{r-2} \subset D_r$ *and the roots are* $\{1, 3, \ldots, 2(r - r_j) - 3, r + n - r_j - 1\}$ *if* Π_j *is* W*-conjugate to a set of simple roots in* $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n} \subset A_{r-2} \subset D_r$.

Proof. The results for A_r and B_r follow immediately from the observation. Similarly for D_r when Π_j is not conjugate to a set of simple roots in $A_{r-2} \subset D_r$. For the case in D_r when Π_j is conjugate to a set of simple roots in $A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n}$, the observation ensures that $\{1, 3, \ldots, 2(r - r_j) - 3\}$ are roots. The remaining root can be determined by noting that the sum of all the roots must equal the number of hyperplanes in X_j. In this particular case, the number of hyperplanes in X_j is seen to be $(r-r_j)(r-r_j-1)+n$, whence the last root must equal $r+n-r_j-1$. □ □

6. Euler Characteristic Computation

The motivation for defining U_t came from studying certain fixed point varieties on an affine flag manifold. We now describe this situation and compute the Euler characteristic of these varieties.

Let G be a simple, simply connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$ of the same type as Φ. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{b}$ be the Borel subalgebra containing $\mathfrak{h}$ corresponding to the simple roots in Π. Let $\mathfrak{b}^{\text{opp}}$ be the opposite Borel subalgebra. For each $\phi \in \Phi$ choose a generator e_ϕ for the corresponding root space. Denote by $\text{ht}(\phi)$, $\phi \in \Phi$, the sum of the coefficients of ϕ when expressed as a linear combination of simple roots in Π. Let $\Phi_k = \{\phi \in \Phi |\, \text{ht}(\phi) = k\}$.

Let $F = \mathbf{C}((\epsilon))$ and $A = \mathbf{C}[[\epsilon]]$. Let $\hat{G} = G(F)$, $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{C}} F$, and $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$. Let $p : \mathfrak{g}_A \to \mathfrak{g}$ be evaluation at $\epsilon = 0$ and define $\hat{\mathfrak{b}}_0 = p^{-1}(\mathfrak{b}^{\text{opp}})$. The $\mathbf{C}$-Lie subalgebras of $\hat{\mathfrak{g}}$ (other than $\hat{\mathfrak{g}}$ itself) which contain $\hat{\mathfrak{b}}_0$ are in bijection with proper subsets J of $I = \{s_0, s_1, \ldots, s_r\}$. Let $\hat{\mathfrak{b}}_0^J$ denote the subalgebra corresponding to the proper subset J.

Let $\hat{\mathcal{B}}^J$ denote the set of $\hat{G}$-conjugates of $\hat{\mathfrak{b}}_0^J$. This set can be given the topology of an increasing union of complex projective varieties. We refer to it as the partial affine flag manifold of type J (J is always assumed to be a proper subset of I). In the case J is the empty set, we write $\hat{\mathcal{B}}$ for $\hat{\mathcal{B}}^J$ and call it the affine flag manifold. There is a natural projection π^J from $\hat{\mathcal{B}}$ to $\hat{\mathcal{B}}^J$ for any J with fiber equal to a finite dimensional partial flag manifold.

For any $n \in \hat{\mathfrak{g}}$ let $\hat{\mathcal{B}}_n^J$ be the subset of $\hat{\mathcal{B}}^J$ consisting of subalgebras which contain n.

In [F], Fan gives the construction of Lusztig of a set of regular, semisimple, nil-elliptic elements of Coxeter type (for definitions see [KL]) depending on a natural number t. We give this construction (up to conjugation by the longest element in the Weyl group). Write $t = ah + b$ where $0 \leq b < h$. Define

$$n_t = \epsilon^a(\epsilon \sum_{\phi \in \Phi_{h-b}} e_\phi + \sum_{\phi \in \Phi_{-b}} e_\phi).$$

When t is relatively prime to the Coxeter number h, the fixed point space $\hat{\mathcal{B}}_{n_t}^J$ is a complex projective variety.

Let χ denote the Euler characteristic. In [LS], $\chi(\hat{\mathcal{B}}_{n_t}^J)$ is computed in type A_r for two partial affine flag manifolds and in [F], $\chi(\hat{\mathcal{B}}_{n_t})$ is computed in all types. We now give a proposition which computes $\chi(\hat{\mathcal{B}}_{n_t}^J)$ in all cases. D. S. Sage proved this proposition for the classical groups, but the combinatorics in his proofs is different [Sa].

Proposition 6.1. *Let j be the cardinality of J. When t is prime to h,*

$$\chi(\hat{\mathcal{B}}^J_{n_t}) = \frac{(t+m_1)(t+m_2)...(t+m_j)t^{r-j}}{|W^J|}$$

where $m_1, \dots, m_j$ are the exponents of W^J.

Before giving the proof, we want to be able to access the results of the previous sections. So we need to introduce some more ideas.

Our main tool is a $\mathbf{C}^*$ action on $\hat{\mathfrak{g}}$ which gives a $\mathbf{C}^*$ action on $\hat{\mathcal{B}}$. We recall the construction given in [F]. Let T be the maximal torus in G with Lie algebra $\mathfrak{h}$. Let $\hat{\rho} : \mathbf{C}^* \to T$ be the one parameter subgroup of T such that $\alpha(\hat{\rho}(\lambda)) = \lambda^{-2}$ for all $\alpha \in \Phi^+$ (viewing the roots as characters of T). Denote by S the image of $\hat{\rho}$ in T. Let $\mathbf{C}^*$ act on $\mathfrak{g}$ through conjugation by S. Let $\lambda \in \mathbf{C}^*$ act on F by the rule $\lambda \circ f(\epsilon) = f(\lambda^{2h}\epsilon)$. Define the action of $\mathbf{C}^*$ on $\hat{\mathfrak{g}}$ by extending $\mathbf{C}$-linearly. Note that $\lambda \circ \epsilon^k e_\phi = \lambda^{2(hk - \mathrm{ht}(\phi))}\epsilon^k e_\phi$.

This $\mathbf{C}^*$ action has a number of key properties. First, it defines an (algebraic) action on any partial affine flag manifold $\hat{\mathcal{B}}^J$ and preserves the fixed point space $\hat{\mathcal{B}}^J_{n_t}$. Second, the fixed points of the $\mathbf{C}^*$ action on $\hat{\mathcal{B}}$ are of the form $w\hat{\mathfrak{b}}_0 w^{-1}$ where $w \in W_a$ (we do not distinguish here between elements in W_a and their representatives in $\hat{G}$ when there is no confusion).

Let $\hat{G}^J = \{g \in \hat{G} | g\hat{\mathfrak{b}}^J_0 g^{-1} = \hat{\mathfrak{b}}^J_0\}$. The quotient of $\hat{G}^J$ by its prounipotent radical is G^J which is a connected reductive algebraic group over $\mathbf{C}$. Let $\mathfrak{g}^J$ be the Lie algebra of G^J and let $p^J : \hat{\mathfrak{b}}^J_0 \to \mathfrak{g}^J$ be the canonical map. The last property of the $\mathbf{C}^*$ action needed in the last section is that $p^J(\lambda \circ n)$ is G^J-conjugate to $p^J(n)$ for $\lambda \in \mathbf{C}^*$ and $n \in \hat{\mathfrak{b}}^J_0$.

We now give another way to define the permutation representation U_t needed in the proof. Let $I_t = \{w \in W_a | \, wD_1 \subset D_t\}$. Since D_t is a fundamental domain for $W_t = W \ltimes tL$, we see that I_t is a set of right coset representatives for W_t in W_a. On the other hand, a set of coset representatives for tL in L also gives a set of right coset representatives for W_t in W_a. Hence there is a natural bijection between I_t and $S_t = L/tL$ which sends $w \in I_t$ to the element in S_t which represents the same right coset of W_t. Explicitly, the map sends w to $-w^{-1}(0)$. Furthermore, W_a acts on I_t by the inverse of right multiplication on the set of right cosets. This action, expressed in terms of S_t, is just induced from the action of W_a on L. As such it is the action we have been discussing.

Proof of Proposition 6.1. The $\mathbf{C}^*$ action preserves $\hat{\mathcal{B}}^J_{n_t}$. Let $\mathcal{F}$ be the points of $\hat{\mathcal{B}}^J_{n_t}$ fixed under the $\mathbf{C}^*$ action. Let χ_c denote the Euler characteristic with compact support. We have $\chi(\hat{\mathcal{B}}^J_{n_t}) = \chi_c(\hat{\mathcal{B}}^J_{n_t})$ since $\hat{\mathcal{B}}^J_{n_t}$ is projective and $\chi_c(\hat{\mathcal{B}}^J_{n_t}) = \chi_c(\mathcal{F})$ (see [B–B]). So our calculation reduces to determining the cardinality of the finite set $\mathcal{F}$.

In general $\mathcal{F} = \{w\hat{\mathfrak{b}}^J_0 w^{-1} | \, w \in W_a \text{ and } n_t \in w\hat{\mathfrak{b}}^J_0 w^{-1}\}$. But this set is in bijection with the set $\{w \in W_a/W^J | \, n_t \in w\hat{\mathfrak{b}}_0 w^{-1}\}$. This takes into account the fact that $w\hat{\mathfrak{b}}_0^J w^{-1}$ stays the same if w is modified by an element of W^J on the right.

Define D^a to be

$$\{u \in V | (u, \alpha) \geq -a \text{ for } \alpha \in \Phi_b \text{ and } (u, \alpha) \geq -a-1 \text{ for } \alpha \in \Phi_{b-h}\}.$$

A result in [F] implies the existence of $\tilde{w} \in W_a$ such that $\tilde{w}(D^a) = D_t$.

Now a calculation shows that $n_t \in w\hat{\mathfrak{b}}_0 w^{-1}$ if and only if $wD_1 \subset D^a$. And this is the case if and only if $\tilde{w}w \in I_t$. So the cardinality of the set $\{w \in W_a/W^J | \, n_t \in w\hat{\mathfrak{b}}_0 w^{-1}\}$ is just the number of orbits of W^J acting on I_t on the right. Under our bijection with S_t, this is the number of W^J-orbits on S_t.

In general, the number of orbits of a finite group H acting on a set is given by

$$\frac{1}{|H|} \sum_{h \in H} \sigma(h)$$

where $\sigma(h)$ is the number of fixed points of h on the set. By Proposition 3.9 we thus have

$$\begin{aligned} \chi(\hat{\mathcal{B}}^J_{n_t}) &= \frac{1}{|W^J|} \sum_{w \in W^J} t^{r-s(w)} \\ &= \frac{t^{r-j}}{|W^J|} \sum_{w \in W^J} t^{j-s(w)}. \end{aligned}$$

But a theorem of Shepard and Todd [ShTo] is that

$$\sum_{w \in W^J} q^{j-s(w)} = (q+m_1)(q+m_2)...(q+m_j),$$

whence the result.

Remark 6.2. We can also view the set S_t as the set of elements of order t in a fixed maximal torus of G. One maps $u \in \Phi^{\vee}$ to $u(\tau) \in T$ where τ is a primitive t-th root of unity. Then the W-orbits of S_t are in bijection with the conjugacy classes of elements of order t in G. This is the viewpoint of D. Peterson who earlier computed the cardinality of the conjugacy classes of elements of order t when t is prime to the Coxeter number.

7. Action of W^J on the cohomology of $\hat{\mathcal{B}}_{n_t}$

In [L1] an action of the affine Weyl group is defined on the homology of $\hat{\mathcal{B}}_n$ for any $n \in \hat{\mathfrak{g}}$ such that $\hat{\mathcal{B}}_n \neq \varnothing$ using the theory of perverse sheaves. Here we are interested in the case where $\hat{\mathcal{B}}_n$ is projective and we will take our action of W_a to be on the cohomology of $\hat{\mathcal{B}}_n$ with complex coefficients. As before, we choose a finite Weyl subgroup W^J of W_a. Here we compute explicitly the virtual representation of W^J on

$$H^*(\hat{\mathcal{B}}_{n_t}) = \sum_{i \geq 0} (-1)^i H^i(\hat{\mathcal{B}}_{n_t})$$

when t is prime to h. Assume from now on that t is prime to h so that $\hat{\mathcal{B}}_{n_t}$ is projective.

Let $\mathcal{B}^J$ be the flag variety of G^J which we identify with the set of Borel subalgebras of $\mathfrak{g}^J$. For any $N \in \mathfrak{g}^J$, let $\mathcal{B}^J_N$ be the subvariety of $\mathcal{B}^J$ consisting of Borel subalgebras containing N. There is a Springer representation of W^J on $H^*(\mathcal{B}^J_N)$. It is known that the cohomology of $\mathcal{B}^J_N$ vanishes in odd degrees (see [DLP]). The next result can be found in [AL].

Theorem 7.1. (Alvis–Lusztig) *Let $N \in \mathfrak{g}^J$ be a nilpotent element, regular in a Levi subalgebra of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}^J$. Then the representation of W^J on $H^*(\mathcal{B}^J_N)$ is isomorphic to $\mathrm{Ind}_P^{W^J}(1)$ where P is the parabolic subgroup of W^J corresponding to $\mathfrak{p}$.*

Now let $N \in \mathfrak{g}^J$ be a nilpotent element. Define $Y^J_N \subset \hat{\mathcal{B}}^J_{n_t}$ to be the set of $g\hat{\mathfrak{b}}^J_0 g^{-1} \in \hat{\mathcal{B}}^J_{n_t}$ where $g \in \hat{G}$ such that $p^J(g^{-1}n_t g)$ is G^J-conjugate to N (this is well defined). The Y^J_N are locally closed subvarieties of $\hat{\mathcal{B}}^J_{n_t}$ and we have $\hat{\mathcal{B}}^J_{n_t} = \cup Y^J_N$, where the union is over a set of representatives of the nilpotent orbits in $\mathfrak{g}^J$ (see [KL]). The closure of any Y^J_N is a union of various $Y^J_{N'}$ and so the Y^J_N give an α-partition of $\hat{\mathcal{B}}^J_{n_t}$ in the language of [DLP]. Recall that π^J is the natural map from $\hat{\mathcal{B}}$ to $\hat{\mathcal{B}}^J$ with fiber isomorphic to $\mathcal{B}^J$. Thus $Y_N := (\pi^J)^{-1}(Y^J_N)$ is a bundle over Y^J_N with fiber $\mathcal{B}^J_N$.

The next theorem is due to G. Lusztig. To prove it, we will need the following fact about derived categories. Let X be an algebraic variety and let $\mathcal{D}(X)$ be the bounded derived category of constructible $\mathbf{C}$-sheaves on X. Let U be an open subset of X with inclusion $i : U \to X$ and let $Z = X - U$ with inclusion $j : Z \to X$. Let $K \in \mathcal{D}(X)$ and let $K' = i_! i^* K$ and $K'' = j_* j^* K$. Then there is a distinguished triangle $K' \to K \to K'' \to K'[1]$ and any endomorphism of K induces an endomorphism of the distinguished triangle. Therefore, any endomorphism of K induces an endomorphism of the long exact sequence

$$\cdots \to H^i(X, K') \to H^i(X, K) \to H^i(X, K'') \to H^{i+1}(X, K') \to \dots.$$

Theorem 7.2. (Lusztig) *The representation of W^J on $H^*(\hat{\mathcal{B}}_{n_t})$ is isomorphic as a virtual W^J-module to*

$$\sum \chi_c(Y^J_N) H^*(\mathcal{B}^J_N)$$

where the sum is over a set of representatives of the nilpotent orbits in $\mathfrak{g}^J$.

Proof. First, we will establish an induced action of W^J on each $H^*_c(Y_N)$ and prove that $H^*(\hat{\mathcal{B}}_{n_t}) = \sum H^*_c(Y_N)$ as virtual representations. Using the theory of perverse sheaves, Lusztig [L1] defines an action of W^J on $K = (\pi^J)_!(\mathbf{C})$ where $\mathbf{C}$ is the constant sheaf on $\hat{\mathcal{B}}_{n_t}$. Since we are only interested in virtual representations and the Y^J_N give an α-partition of $\hat{\mathcal{B}}^J_{n_t}$, we can repeatedly apply the naturality of the above long exact sequence and we get

$$H^*(\hat{\mathcal{B}}^J_{n_t}, K) = \sum H^*(\hat{\mathcal{B}}^J_{n_t}, (i_N)_!(i_N)^*K),$$

where $i_N : Y^J_N \to \hat{\mathcal{B}}^J_{n_t}$ denotes the inclusion. But we have $H^*(\hat{\mathcal{B}}^J_{n_t}, K) = H^*(\hat{\mathcal{B}}_{n_t}, \mathbf{C})$ and $H^*(\hat{\mathcal{B}}^J_{n_t}, (i_N)_!(i_N)^*K) = H^*_c(Y_N, \mathbf{C})$. This establishes the induced action of W^J on $H^*_c(Y_N)$ and the equality $H^*(\hat{\mathcal{B}}_{n_t}) = \sum H^*_c(Y_N)$.

To complete the proof, we must show that $H^*_c(Y_N) = \chi_c(Y^J_N)H^*(\mathcal{B}^J_N)$ as virtual representations of W^J. Now restrict π^J to Y_N and let K refer to $(\pi^J)_!(\mathbf{C})$ where $\mathbf{C}$ is now the constant sheaf on Y_N. The W^J-action on $H^*_c(Y_N)$ from the previous paragraph comes from the action of W^J on K. There is a spectral sequence with $E^{p,q}_2 = H^p_c(Y^J_N, \mathcal{H}^q(K))$ converging to $H^{p+q}_c(Y^J_N, K)$, where $\mathcal{H}$ denotes the cohomology sheaf, and this spectral sequence is functorial in K. But since we are interested in virtual vector spaces, the spectral sequence is degenerate and we have

$$\sum_m (-1)^m \sum_{p+q=m} H^p_c(Y^J_N, \mathcal{H}^q(K)) = H^*_c(Y^J_N, K)$$

as virtual W^J-modules. Now the right side above equals $H^*_c(Y_N)$ with its W^J-action. Because Y_N is a fiber bundle over Y^J_N with fiber $\mathcal{B}^J_N$, the term $H^p_c(Y^J_N, \mathcal{H}^q(K))$ is isomorphic to $H^p_c(Y^J_N) \otimes H^q(\mathcal{B}^J_N)$ with the induced W^J-action on $H^q(\mathcal{B}^J_N)$. So the left side above equals $\chi_c(Y^J_N)H^*(\mathcal{B}^J_N)$. □ □

Putting together these two theorems with our previous work we can conclude

Theorem 7.3. *The virtual representation of W^J on $H^*(\hat{\mathcal{B}}_{n_t})$ is isomorphic to the restriction of U_t to W^J.*

Proof. The $\mathbf{C}^*$ action on $\hat{\mathcal{B}}^J_{n_t}$ preserves the subvarieties Y^J_N. Let $\mathcal{F}_N$ denote the fixed points of $\mathbf{C}^*$ on Y^J_N. We have $\chi_c(Y^J_N) = \chi_c(\mathcal{F}_N)$. So to compute $\chi_c(Y^J_N)$ we only have to determine the cardinality of $\mathcal{F}_N$.

Let $N \in \mathfrak{g}$ be a nilpotent element. Choose $w_1\hat{\mathfrak{b}}^J_0 w_1^{-1} \in \mathcal{F}_N$. We may assume that $n_t \in w_1\hat{\mathfrak{b}}_0 w_1^{-1}$ by modifying w_1 by an element of W^J on the right.

Consider again the map $\pi^J : \hat{\mathcal{B}}_{n_t} \to \hat{\mathcal{B}}^J_{n_t}$. The fiber above the point $w_1\hat{\mathfrak{b}}^J_0 w_1^{-1}$ is isomorphic to $\mathcal{B}^J_N$. Since π^J is $\mathbf{C}^*$-equivariant, we get a $\mathbf{C}^*$ action on $\mathcal{B}^J_N$ with fixed points that we identify with $\{w \in W^J | N \in w\mathfrak{b}_0 w^{-1}\}$. On the other hand, the analysis from the previous section shows that this set is just a set of right coset representatives for the stabilizer in W^J of $\tilde{w}w_1 \in I_t$. This stabilizer is a parabolic subgroup P of W^J by Proposition 4.1. Now an easy argument shows that if the set $\{w \in W^J | N \in w\mathfrak{b}_0 w^{-1}\}$ is a set of right coset representatives of a parabolic subgroup P of W^J, then N must be conjugate to a nilpotent which is regular in a Levi factor of a parabolic subalgebra in $\mathfrak{g}^J$ corresponding to P.

Thus when N is conjugate to a regular nilpotent in a Levi factor of a parabolic subalgebra in $\mathfrak{g}^J$ corresponding to P, $\chi_c(Y^J_N)$ equals the number of orbits in I_t (or S_t) with stabilizer conjugate to P. Moreover, if N is not such a nilpotent, then $\chi_c(Y_N^J) = 0$.

Putting this argument together with the previous two theorems and our analysis of the representation U_t yields the theorem. □ □

Remark 7.4. The virtual representation of W_a on $H^*(\hat{\mathcal{B}}_{n_t})$ is not isomorphic to U_t. This can be seen in A_1 with $t = 3$. Here the variety $\hat{\mathcal{B}}_{n_t}$ is two complex projective lines joined at a point. The representation of W_a on the cohomology of this variety is not completely reducible (using results from [Ka]), whereas the representation U_t is always completely reducible under W_a.

Remark 7.5. In type A_r, Lusztig and Smelt have shown that $\hat{\mathcal{B}}_{n_t}$ has no odd cohomology [LS].

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