CONJECTURES FOR SMALL REPRESENTATIONS OF THE EXCEPTIONAL GROUPS

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ABSTRACT. In this note, G is a connected, simple, adjoint algebraic group of exceptional type over the complex numbers, i.e. of type G_2, F_4, or E_n ($n = 6, 7, 8$). Let V be a small representation for G. We state some conjectures concerning the occurrences of V in the G-module of functions on covers of nilpotent orbits and the occurrences of $(V^T)^\vee$ (the dual Weyl group representation) in Springer representations associated to the dual group $G^\vee$.

1. NOTATION

Let G be a connected, simple, adjoint algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. We assume G is of type G_2, F_4, or E_n ($n = 6, 7, 8$). Recall (after Broer [Br1]) that a small representation for G is a highest weight representation V_λ for which twice a root is not a weight. The possibilities for λ are ω_1, ω_2 for G_2; $\omega_1, \omega_3, \omega_4$ for F_4; $\omega_6, \omega_1 + \omega_5, \omega_3, \omega_1 + \omega_2, 3\omega_1$ for E_6 (up to outer automorphism); $\omega_1, \omega_5, 2\omega_6, \omega_2, \omega_6 + \omega_7$ for E_7; $\omega_7, \omega_1, \omega_6, \omega_8$ for E_8. Here ω_i denotes the i-th fundamental weight where we number in Dynkin's fashion except we flip about the vertical axis in type E_8.

If $x \in \mathfrak{g}$, let $\mathcal{O}_x$ denote the G-orbit in $\mathfrak{g}$ through x. Fix a maximal torus T of G and let $W := N_G(T)/T$ be the Weyl group.

2. CONJECTURES

The first conjecture relates functions on a cover of a non-special nilpotent orbit to functions on the universal cover of the associated special nilpotent orbit.

Let $\mathcal{O}_y$ be a nilpotent orbit in $\mathfrak{g}$ which is not special. Then, by Spaltenstein, there is a well-defined special nilpotent orbit $\mathcal{O}_x$ in $\mathfrak{g}$ associated to $\mathcal{O}_y$. It is defined by the property that $\mathcal{O}_y$ is contained in the closure $\overline{\mathcal{O}}_x$ of $\mathcal{O}_x$ and $\mathcal{O}_y$ is not contained in the closure of any other special orbit contained in $\overline{\mathcal{O}}_x$ (see [Lu]).

Let $A(x) = G_x/(G_x)^0$ and let $\bar{A}(x)$ denote Lusztig's canonical quotient of $A(x)$. We will sometimes refer to nilpotent orbits by their Bala-Carter notation (see [Ca]). In all cases, except when $y = A_7$ in type E_8, we have $A(x) = \bar{A}(x)$ (we are still assuming y is non-special). When $y = A_7$ and hence $x = E_8(b_6)$, we have that $A(x) = S_3$ and $\bar{A}(x) = S_2$. Hence in all

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cases, $\bar{A}(x)$ can be viewed as a Coxeter group. In [Lu], Lusztig has attached to the pair (y, x) a parabolic subgroup $H = H(y, x)$ of $\bar{A}(x)$. Since we are assuming y is not special, we have $H \neq 1$. In the exceptional groups, Lusztig has observed that $A(y) = N_{\bar{A}(x)}(H)/H$, where $N_{\bar{A}(x)}(H)$ denotes the normalizer of H in $\bar{A}(x)$.

Let $\tilde{\mathcal{O}}_x$ denote the largest cover of the orbit $\mathcal{O}_x$ with a G-action and let $R^*(-)$ denote the graded global functions on either of these varieties. The group $A(x)$ acts on $\tilde{\mathcal{O}}_x$ inducing a representation of $A(x)$ on $R^i(\tilde{\mathcal{O}}_x)$. This action commutes with the G-action.

On the level of small representations, we conjecture that $R^i(\tilde{\mathcal{O}}_y)$ is completely determined by $R^i(\tilde{\mathcal{O}}_x)$. We fix some more notation. For a group K, let $K^\wedge$ denote its set of irreducible representations and for K-representations V and W let $[V : W]_K$ denote the multiplicity of W in V. If $\psi \in \bar{A}(x)^\wedge$, we may consider ψ as a representation of $A(x)$ by pullback and we may consider ψ^H (the H-invariants in ψ) as a representation of $A(y) = N_{\bar{A}(x)}(H)/H$. We can now state our first conjecture.

Conjecture 2.1. *Let $V = V_\lambda$ be a small representation and let $\phi \in A(y)^\wedge$. We have*

$$[R^i(\tilde{\mathcal{O}}_y) : V \otimes \phi]_{G \times A(y)} = \sum_{\psi \in \bar{A}(x)^\wedge} [R^i(\tilde{\mathcal{O}}_x) : V \otimes \psi]_{G \times A(x)} [\psi^H : \phi]_{A(y)}.$$

Remark 2.2. While there are Kostant-like multiplicity formulas for $R^i(\mathcal{O}_x)$, there are no known formulas for $R^i(\tilde{\mathcal{O}}_x)$ in general. In many cases, we can verify the conjecture by using Lusztig-Spaltenstein induction which yield formulas (after the work of Borho-Kraft and McGovern) for certain orbit covers. In the general case, assuming the conjectures in the author's thesis [So1] which propose formulas in general, we have checked the conjecture by computer for all cases in G_2, F_4, E_6; for ω_1, ω_5 in E_7; and for ω_7 in E_8. In E_7, we have checked the conjecture whenever ϕ is non-trivial. At the present time, many (but not all) of the formulas in [So1] have been proven for the exceptional groups [So2].

We now observe that the conjecture helps explain why the closures of certain non-special nilpotent orbits fail to be normal varieties. Assume that we choose $\phi = 1$ in the above conjecture. Then the left side of the conjecture just yields the multiplicity of V_λ in $R^i(\mathcal{O}_y)$.

Assume furthermore that $H = H(y, x) = \bar{A}(x)$ (i.e., $\mathcal{O}_y$ is the smallest non-special orbit attached to $\mathcal{O}_x$), then the only non-zero term in the sum on the right side occurs when ψ is the trivial representation. Hence, the right side just yields the multiplicity of V_λ in $R^i(\mathcal{O}_x)$. Thus in this situation the multiplicity of a small representation in $R^i(\mathcal{O}_y)$ is the same as its multiplicity in $R^i(\mathcal{O}_x)$. This perhaps suggests that $\overline{\mathcal{O}}_y$ is normal if and only if $\overline{\mathcal{O}}_x$ is normal in the case where $H = \bar{A}(x)$. This is known to be true in G_2 and F_4 for example.

On the other hand, when $\bar{A}(x) = S_3, S_4$, or S_5 and $H \neq \bar{A}(x)$, then the right side will yield more than the multiplicity of V_λ in $R^i(\mathcal{O}_x)$ for at least one choice of λ. In this case, $\overline{\mathcal{O}}_y$ can not possibly be normal since $R^i(\mathcal{O}_y)$ is not a quotient as G-module of $R^i(\mathcal{O}_x)$, a necessary condition for the normality of $\overline{\mathcal{O}}_y$.

Conjecture 1 has an analog for Green functions on the dual group. In this situation, it can be verified because we have the tables of Beynon and Spaltenstein [BS] and Shoji [Sh].

Let $G^\vee$ be the dual group of G and let $\mathfrak{g}^\vee$ be its Lie algebra. The flag variety $\mathcal{B}^\vee$ of $G^\vee$ can be identified with the variety of all Borel subalgebras in $\mathfrak{g}^\vee$. Then if $x \in \mathfrak{g}^\vee$, the corresponding fixed point subvariety of $\mathcal{B}^\vee$ is

$$\mathcal{B}^\vee_x = \{\mathfrak{b} \in \mathcal{B}^\vee | x \in \mathfrak{b}\}.$$

The Weyl group $W^\vee$ of $G^\vee$ acts on the cohomology $\mathrm{H}^*(\mathcal{B}^\vee_x)$ via Springer representation and so does $A(x)$ and these actions commute. The Weyl group of G acts on V^T. Let $(V^T)^\vee$ be the $W^\vee$-module dual to the W-module V^T.

Let y be a non-special nilpotent element in $\mathfrak{g}^\vee$ and x an associated special nilpotent element. As above, let V be a small representation of G and let $\phi \in A(y)^\wedge$.

Proposition 2.3.

$$[H^i(\mathcal{B}^\vee_y) : (V^T)^\vee \otimes \phi]_{W^\vee \times A(y)} = \sum_{\psi \in \bar{A}(x)^\wedge} [H^i(\mathcal{B}^\vee_x) : (V^T)^\vee \otimes \psi]_{W^\vee \times A(x)} [\psi^H : \phi]_{A(y)}.$$

Proof. The multiplicity of an irreducible representation of $W^\vee$ in the ϕ-isotypic component of $H^i(\mathcal{B}^\vee_y)$ can be read off the tables of Shoji in type F_4 [Sh] and Beynon and Spaltenstein in types E_6, E_7, E_8 [BS]. We used these tables to verify the above identity by hand in those cases (the case of G_2 is straightforward). In fact, the identity holds for each irreducible constituent of $(V^T)^\vee$. □

This proposition is very similar to an observation of Lusztig [Lu, Proposition 0.7], but only overlaps that observation in a few cases. We now put the ring of functions picture together with the Springer picture in the following conjecture. Let d denote Spalentstein's order-reversing involution from the set of special nilpotent orbits of $\mathfrak{g}^\vee$ to the special nilpotent orbits of $\mathfrak{g}$.

Conjecture 2.4. *Let x be a special nilpotent element in $\mathfrak{g}^\vee$. Let $\psi \in A(x)^\wedge$ occur in the Springer correspondence. Then*

$$[H^{2i}(\mathcal{B}^\vee_x) : (V^T)^\vee \otimes \psi]_{W^\vee \times A(x)} = [R^i(\tilde{\mathcal{O}}_{d(x)}) : V \otimes \psi]_{G \times A(d(x))}.$$

We have to be careful in the above statement because $A(x)$ and $A(d(x))$ do not in general coincide. If they do coincide or if $A(x) = 1$, the conjecture makes perfect sense. If $A(x) = S_2$ and $A(d(x)) = 1$, then the conjecture is to be interpreted as saying $[H^i(\mathcal{B}^\vee_x) : (V^T)^\vee \otimes \psi] = 0$ when ψ is the sign representation of $A(x)$. In other words, there is nothing on the right hand side of the conjecture. The only remaining cases are in type E_8 when $x = E_8(b_6)$ and $d(x) = D_4(a_1) + A_2$ or instead $x = D_4(a_1) + A_2$ and $d(x) = E_8(b_6)$. In both cases we can match up trivial representations and sign representations. When $x = E_8(b_6)$, we are saying that $[H^i(\mathcal{B}_x^\vee) : (V^T)^\vee \otimes \psi] = 0$ when ψ is the 2-dimensional representation of $A(x)$.

We have checked this conjecture for all of the cases where the right side can be handled by a computer (again assuming the conjectures in [So1] mentioned in the first remark). The cases checked include all small representations in G_2, F_4, E_6; ω_1, ω_5 in E_7; and ω_7 in E_8. In E_7, we have checked the conjecture whenever ϕ is non-trivial.

Remark 2.5. The last conjecture and the proposition together imply a conjecture of Broer for the exceptional groups. In Broer's picture y is a nilpotent which is regular in a Levi factor and x is an associated special nilpotent element. Since y is regular in a Levi factor of $\mathfrak{g}^\vee$, we have that $d(x)$ is a Richardson element for the dual Levi subalgebra in $\mathfrak{g}$. Moreover, by a purely empirical observation, the subgroup of $A(d(x))$ obtained by induction (which measures the failure of birationality of the moment map from the appropriate cotangent bundle) coincides with the subgroup $H(y,x)$ of $A(x)$ that Lusztig attaches to y and x. However, we have observed that when y is not regular in a Levi factor, it can happen that the subgroup of $A(d(x))$ obtained by induction is not in general the same subgroup as $H(y,x)$ of $A(x)$. For example, this occurs in type F_4.

The special case of Broer's conjectures for the adjoint representation is known after the work of Broer [Br2], [Br3]; Sommers-Trapa [ST]; and Douglass [Do].

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