LOCAL SYSTEMS ON NILPOTENT ORBITS AND WEIGHTED DYNKIN DIAGRAMS

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Abstract. We study the Lusztig-Vogan bijection for the case of a local system. We compute the bijection explicitly in type A for a local system and then show that the dominant weights obtained for different local systems on the same orbit are related in a manner made precise in the paper. We also give a conjecture (putatively valid for all groups) detailing how the weighted Dynkin diagram for a nilpotent orbit in the dual Lie algebra should arise under the bijection.

1. Introduction

Let $G$ be a reductive algebraic group over the complex numbers, $B$ a Borel subgroup of $G$, and $T$ a maximal torus of $B$. We denote by $\Lambda = \Lambda(G)$ the weight lattice of $G$ with respect to $T$, and by $\Lambda_+ = \Lambda_+(G)$ the set of dominant weights with respect to the positive roots defined by $B$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $\mathcal{N}$ denote the nilpotent cone in $\mathfrak{g}$.

Now, let $e \in \mathcal{N}$ be a nilpotent element, and let $\mathcal{O}_e$ be the orbit of $e$ in $\mathfrak{g}$ under the adjoint action of $G$. We write $\mathcal{O}^e$ for the centralizer of $e$ in $G$. Let $\mathcal{N}_o$ denote the set of nilpotent orbits in $\mathfrak{g}$, and $\mathcal{N}_{o,r}$ the set of $G$-conjugacy classes of pairs

$$\{(e, \tau) \mid e \in \mathcal{N} \text{ and } \tau \text{ an irreducible rational representation of } G^e\}.$$ 

Lusztig [10] conjectured the existence of a bijection $\mathcal{N}_{o,r} \leftrightarrow \Lambda_+$ using his work on cells in affine Weyl groups. From the point of view of Harish-Chandra modules, Vogan also conjectured a bijection between $\mathcal{N}_{o,r}$ and $\Lambda_+$. Such a bijection has been established by Bezrukavnikov in two preprints (the bijections in each preprint are conjecturally the same) [2], [4]. Bezrukavnikov’s second bijection is closely related to Ostrik’s conjectural description of the bijection [13] (see also [5]). In the case of $G = GL(n, \mathbb{C})$, the first author [1] described an explicit combinatorial bijection between $\mathcal{N}_{o,r}$ and $\Lambda_+$ from the Harish-Chandra module perspective. At present, it is not known how all of these bijections are related (Bezrukavnikov’s two candidates; Ostrik’s conjectural candidate; and the first author’s candidate in type $A$). In this paper, we work in the context of [1], which we now review.

Let $K_G(\mathcal{N})$ denote the Grothendieck group of $G$-equivariant coherent sheaves on $\mathcal{N}$. On the one hand, $K_G(\mathcal{N})$ has a natural basis indexed by elements of $\Lambda_+$, denoted $AJ(\lambda)$ in [13] (but unlike in [13], we are not utilizing the $\mathbb{C}^*$ action on $\mathcal{N}$ here). The algebraic global sections of $AJ(\lambda)$ are isomorphic as a $G$-module to $Ind^G_T \lambda$. Thus, the space of global

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\footnote{Bezrukavnikov has announced that his bijections coincide with each other and Ostrik’s [3]. We also note that Xi has established a bijection in type $A$ [16].}
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sections of any element $F \in K_{G}(N)$ is given as a $G$-module by a unique expression of the form

$$\Gamma(N, F) = \sum_{\lambda \in \Lambda_{+}} m_{\lambda} \text{Ind}_{G}^{\mathbb{C}} \lambda,$$

where the $m_{\lambda} \in \mathbb{Z}$, and $m_{\lambda} \neq 0$ for only finitely many $\lambda$. (This fact was communicated to us by Vogan.)

Let us now fix a pair $(e, \tau) \in N_{o,r}$. We want to consider all elements $F \in K_{G}(N)$ whose support is contained in $O_{e}$, and whose restriction to $O_{e}$ is the vector bundle arising from $\tau$. For each such $F$, there is at least one $\lambda$ of maximal length occurring in the expression (1) (where we have fixed a $W$-invariant positive-definite symmetric bilinear form on the real span of $\Lambda$, so that we can speak of the length of a weight of $G$). Define $\gamma : N_{o,r} \rightarrow \Lambda_{+}$ by

$$\gamma(e, \tau) = \text{the smallest largest } \lambda, \text{ over all possible choices of } F.$$

The following conjecture was made in [1]; moreover, it was proved in the case of $G = GL(n, \mathbb{C})$ in op. cit.

**Conjecture 1.1** ([1]). The map $\gamma$ is well-defined and is a bijection. Moreover, there is a basis $\{M(e, \tau)\}$ for $K_{G}(N)$, indexed by $N_{o,r}$, such that

1. the support of $M(e, \tau)$ is $O_{e}$,
2. the restriction of $M(e, \tau)$ to $O_{e}$ is the locally free sheaf arising from $\tau$, and
3. there is an upper-triangular relationship between this basis and the one indexed by $\Lambda_{+}$:

$$\Gamma(N, M(e, \tau)) = \pm \text{Ind}_{G}^{\mathbb{C}} \gamma(e, \tau) + \sum_{\|\mu\| < \|\gamma(e, \tau)\|} m_{\mu} \text{Ind}_{G}^{\mathbb{C}} \mu.$$

In this paper, we study the problem of computing $\gamma$ when $\tau$ gives rise to a local system on $O_{e}$ (for semisimple groups, this means that $\tau$ is trivial on the identity component of $G_{e}$). Let us denote by $\Lambda_{+}^{\text{loc}}$ the image of $\gamma$ when $\tau$ corresponds to a local system.

Here is an outline of the paper. In section 2, we compute $\Lambda_{+}^{\text{loc}}$ explicitly for $G = GL(n)$. In section 3, we state a precise conjecture, for general $G$, that the Dynkin weights for the Langlands dual group of $G$ are a subset of $\Lambda_{+}^{\text{loc}}$. In section 4, we show how to associate to $\gamma(e, \tau)$ a sub-bundle of the cotangent bundle of $G/B$. Then for $GL(n)$, we are able to prove that when $e$ is in a fixed nilpotent orbit, the cohomologies of the sub-bundles (with coefficients in the structure sheaf) are independent of the local system $\tau$.

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## 2. Computing $\gamma$ for a local system in $GL(n)$

For a partition $d$ of $n$, let $O_{d}$ denote the nilpotent orbit in $\mathfrak{g}l(n)$ indexed by $d$. If $d = [k_{1}^{a_{1}}, k_{2}^{a_{2}}, \ldots, k_{l}^{a_{l}}]$, then we know that $\pi_{1}(O_{d}) \cong \mathbb{Z}/c\mathbb{Z}$, where $c$ is the greatest common divisor of the $k_{i}$’s [6]. Let $G^{d}$ denote the isotropy group of an element in this orbit, and $G_{\text{red}}^{d}$ the reductive part thereof. Following the notation of [6], we have

$$G_{\text{red}}^{d} \cong GL(a_{1})^{k_{1}}_{\Delta} \times \cdots \times GL(a_{l})^{k_{l}}_{\Delta}.$$
Here, \( H_\Delta^n \) denotes the diagonal copy of \( H \) in the product of \( m \) copies of itself. Now, in the case of simply connected semisimple groups, we identify local systems on an orbit with representations of the component group of the centralizer of an element in the orbit. In the setting of \( GL(n) \), we will first produce a list of representations of \( G^d_\text{red} \), and then we will examine their restrictions to \( SL(n) \) to determine the correspondence with local systems.

Let \( A_1 \in GL(a_1) \), \( A_2 \in GL(a_2) \), etc. The set of matrices \( (A_1, \ldots, A_l) \) determines an element \( \tilde{A} \) of

\[
GL(a_1)^{k_1}_\Delta \times \cdots \times GL(a_l)^{k_l}_\Delta;
\]

moreover, we have

\[
\det \tilde{A} = (\det A_1)^{k_1} \cdots (\det A_l)^{k_l}.
\]

Since every \( k_i \) is divisible by \( c \), we can write this as

\[
\det \tilde{A} = ((\det A_1)^{k_1/c} \cdots (\det A_l)^{k_l/c})^c.
\]

Let “\( \text{det}^{1/c} \)” denote the character \( G^d_\text{red} \rightarrow \mathbb{C}^\times \) given by

\[
(\det A_1)^{k_1/c} \cdots (\det A_l)^{k_l/c},
\]

and for any \( p \in \mathbb{Z} \), let \( \text{det}^{p/c} \) denote the \( p \)-th power of this map.

In \( SL(n) \), the isotropy group \( SL(n)^d \) is the subgroup of \( G^d \) consisting of matrices of determinant 1. The function \( \text{det}^{1/c} \) takes only finitely many values when restricted to this subgroup, namely, the \( c \)-th roots of unity. Indeed, the component group of \( SL(n)^d \) is just \( \mathbb{Z}/c\mathbb{Z} \), and the irreducible representations of \( \mathbb{Z}/c\mathbb{Z} \) are just powers of the function \( \text{det}^{1/c} \).

We have, then, that the irreducible representations of \( \pi_1(\mathcal{O}_d) \) come from

\[
\text{det}^{0/c}, \text{det}^{1/c}, \ldots, \text{det}^{(c-1)/c}.
\]

Now we need to compute \( \gamma(\mathcal{O}_d, \text{det}^{p/c}) \) for \( 0 \leq p \leq c-1 \). Suppose that the dual partition of \( d \) is \( d^d = [j_1^{b_1}, \ldots, j_s^{b_s}] \). Following the notation of [1], we let \( L_d \) denote the Levi subgroup

\[
L_d = GL(j_1)^{b_1} \times \cdots \times GL(j_s)^{b_s},
\]

and we let \( P_d \) be the standard (block-upper-triangular) parabolic subgroup whose Levi factor is \( L_d \). Recall that every nilpotent orbit in \( GL(n) \) is Richardson; in particular, \( \mathcal{O}_d \) is Richardson for \( P_d \). We fix a choice of \( e \in \mathcal{O}_d \) such that \( G^d_\text{red} \subset L_d \) and \( G^d \subset P_d \). Explicitly, the embedding of \( G^d \) in \( P_d \) is such that the factor \( GL(a_i)^{k_i}_\Delta \) sits diagonally across the product of the first \( k_i \) factors of \( L_d \). Let \( \rho_d \) denote the half-sum of the positive roots of \( L_d \).

We make the observation that any irreducible representation of \( G^d \) must be trivial on its unipotent radical, hence is completely determined by its restriction to \( G^d_\text{red} \). Therefore, it makes sense to refer to irreducible representations of \( G^d \) by their highest weights. We do this at various points below.

In [1], the computational device of “weight diagrams” was introduced and employed to carry out the computation of \( \gamma \). For our present purposes, however, we are only concerned with a small collection of representations for each orbit, and we do not need to use the cumbersome weight diagrams. Instead, we make use of the following auxiliary result.
Proposition 2.1 ([1], Claim 2.3.1). Let \((\pi_\lambda, V_\lambda)\) denote the irreducible \(L_d\)-representation of highest weight \(\lambda\), regarded as a \(P_d\)-representation by letting the unipotent part of \(P_d\) act trivially, and let \((\tau, W_\tau)\) be an irreducible representation of \(G^d\). Suppose that \(\tau\) occurs as a summand in the restriction of \(\pi_\lambda\) to \(G^d\), and, moreover, that
\[
\|\lambda + 2\rho_d\|^2
\]
is minimal among all irreducible \(P_d\)-representations whose restriction to \(G^d\) contains \(\tau\) as a summand. Then \(\gamma(O_d, \tau) = \lambda + 2\rho_d\), made dominant for \(GL(n)\).

The next proposition describes \(\gamma(O_d, \det^{b/c})\) explicitly in terms of the standard basis. The remainder of the section will be devoted to establishing this proposition. In Section 4, we shall use this explicit description to prove the main result.

Let \(\omega_p = \gamma(O_d, \det^{b/c})\). Writing \(\omega_0\) in the standard basis, let “block \(B_a\)” refer to the collection of coordinate positions which contain the entry \(a\) in \(\omega_0\), for \(a \in \mathbb{Z}\). Let \(\mu_k\) be the multiplicity of \(k\) in \(d^1\). (Thus, if there is some \(j_i\) such that \(j_i = k\), then \(\mu_k = b_i\); otherwise, \(\mu_k = 0\).) Note that every \(b_i\) and every \(\mu_k\) must be a multiple of \(c\).

Proposition 2.2. The length of block \(B_a\) is \(\sum_{k \geq 0} \mu_{k+2k+1}\) when \(a \geq 0\) and the length of \(B_a\) and \(B_{-a}\) are the same. Moreover, if we write the length of block \(B_a\) as \(m_{ac}\), then \(\omega_p\) is obtained by replacing the first \(m_{ac}\) entries in block \(B_a\) with \(a + 1\).

Example 2.3. Consider the orbit labeled by \(d = [6, 3, 3]\) in \(GL(12)\). Then \(c = 3\), \(d^1 = [3, 3, 3, 1, 1, 1]\), and \(\mu_3 = 3\) and \(\mu_1 = 3\).

We illustrate the preceding proposition by listing here all the \(\omega_p\). The first sentence of the proposition describes \(\omega_0\) by giving the lengths of blocks, and the second sentence tells us how to obtain the other \(\omega_p\)’s by modifying \(\omega_0\).

\[
\begin{align*}
\omega_0 &= (2, 2, 2, 2, 0, 0, 0, 0, 0, 0, -2, -2, -2) \\
\omega_1 &= (3, 2, 2, 1, 1, 0, 0, 0, 0, -1, -2, -2) \\
\omega_2 &= (3, 3, 2, 1, 1, 1, 1, 0, 0, -1, -1, -2)
\end{align*}
\]

Proof. Let \(\lambda_p\) be the appropriate dominant weight of \(L_d\) as described in Proposition 2.1, such that \(\omega_p\) is just \(\lambda_p + 2\rho_d\), made dominant for \(GL(n)\). In what follows, we shall be careless about saying “made dominant” every time; the reader should fill in those words wherever appropriate.

Let us begin with the trivial representation of \(\pi_1(O_d)\), namely \(\det^{0/c}\). Whatever \(\lambda_0\) is, it must be dominant for \(L_d\), so that \(\langle \lambda_0, 2\rho_d \rangle \geq 0\). Therefore
\[
\|\lambda_0 + 2\rho_d\|^2 = \|\lambda_0\|^2 + \|2\rho_d\|^2 + 2\langle \lambda_0, 2\rho_d \rangle \geq \|2\rho_d\|^2.
\]
Now 0 is a weight of \(L_d\) with the right restriction to \(G^d\), and taking \(\lambda_0 = 0\) obviously minimizes \(\|\lambda_0 + 2\rho_d\|^2\) (the above inequality becomes an equality). We therefore have \(\omega_0 = 2\rho_d\). What does \(2\rho_d\) look like? For each \(GL(j_i)\) factor of \(L_d\), we get a part of \(2\rho_d\) that looks like
\[
(j_i - 1, j_i - 3, \ldots, 1 - j_i).
\]
Thus, in the total \(2\rho_d\), a particular coordinate \(a\) with \(a \geq 0\) occurs once for each factor \(GL(j_i)\) with \(j_i = a + 2k + 1\) for some \(k\) with \(k \geq 0\). It follows that the length of block \(B_a\)
is precisely \( \sum_{k \geq 0} \mu_{a+2k+1} \), as desired. It is also clear that the length of block \( B_a \) and block \( B_{-a} \) are equal.

Next, we consider the case \( p \neq 0 \). We will consider the first factor \( GL(a_1)^{k_1}_{\Delta} \) of \( G_{\text{red}}^d \) individually; the other factors would be treated identically. The factor \( GL(a_1)^{k_1}_{\Delta} \) of \( G_{\text{red}}^d \) sits diagonally across various \( GL(j_i)^{b_i} \) factors of \( L_d \); indeed, it sits across \( k_1 \) of them. Now, given a weight \( \lambda \) of \( L_d \), we obtain the coordinates of the restriction \( \lambda |_{G_{\text{red}}^d} \) by summing up coordinates of \( \lambda \) according to the diagonal embedding of the factors of \( G_{\text{red}}^d \) in \( L_d \). (See [1] for a detailed account of how to restrict weights from \( L_d \) to \( G_{\text{red}}^d \).) In any case, we add up \( k_1 \) distinct coordinates of \( \lambda \) to obtain each coordinate of the \( GL(a_1)^{k_1}_{\Delta} \) part of the restriction of \( \lambda \). Now, the highest (and only) weight of \( G_{\text{red}}^d \) on the representation \( \det^{b/c} \) is

\[
\left( \begin{array}{c}
pk_1 \\
\vdots \\
pk_1 \\
\end{array} \right)_{GL(a_1)^{k_1}_{\Delta}} \text{ and } \left( \begin{array}{c}
pk_1 \\
\vdots \\
pk_1 \\
\end{array} \right)_{GL(a_1)^{k_1}_{\Delta}}
\]

in particular, every coordinate in the \( GL(a_1)^{k_1}_{\Delta} \) part of the weight is equal to \( pk_1/c \). If we want \( \| \lambda \|^2 \) to be minimal, it is clear that the \( k_1 \) coordinates we add up to obtain this coordinate should consist of \( pk_1/c \) 1’s and \( (c - p)k_1/c \) 0’s.

Repeating this argument for every factor of \( G_{\text{red}}^d \), we see that \( \| \lambda \|^2 \) is minimized if, among all its coordinates, there are \( pn/c \) 1’s and \( (c - p)n/c \) 0’s. But to compute \( \gamma \), we need to minimize \( \| \lambda + 2\rho_d \|^2 \), not \( \| \lambda \|^2 \). We have

\[
\| \lambda + 2\rho_d \|^2 = \| \lambda \|^2 + \| 2\rho_d \|^2 + 2\langle \lambda, 2\rho_d \rangle.
\]

The last term is non-negative, so we will have minimized \( \| \lambda + 2\rho_d \|^2 \) if we can choose \( \lambda \) with \( \| \lambda \|^2 \) minimal and \( \langle \lambda, 2\rho_d \rangle = 0 \). In fact, we can arrange for this to happen.

Among the factors \( GL(j_i)^{b_i} \) of \( L_d \), take the weight \((1, \ldots, 1)\) (i.e., the determinant character) on \( ph_i/c \) of the factors, and \((0, \ldots, 0)\) on the remaining \((c - p)b_i/c \) of them. Concatenating these weights, we obtain a weight \( \lambda_p \) of \( L_d \). It is easy to see that \( \lambda_p \) has the right number of 0’s and 1’s to have the desired restriction to \( G_{\text{red}}^d \) as well as to be of minimal size. Moreover, as promised, we have that \( \langle \lambda_p, 2\rho_d \rangle = 0 \).

Therefore, \( \omega_p = \lambda_p + 2\rho_d \). This looks very similar to \( \omega_0 \), except that in each block \( B_a \), a proportion \( p/c \) of the coordinates have been increased by 1. Thus \( \omega_p \) has precisely the desired form.

\[\square\]

3. A conjecture about Dynkin diagrams

In this section, let \( G \) be connected, simply-connected, and simple. Let \( ^t G, ^t B, ^t T \) be the data of the Langlands dual group corresponding to \( G, B, T \), respectively. Let \( ^t \mathfrak{g}, ^t \mathfrak{b}, ^t \mathfrak{h} \) denote the Lie algebras of \( ^t G, ^t B, ^t T \), respectively. The weights of \( T \) can be identified with the elements \( h \in ^t \mathfrak{h} \) such that \( \alpha^\vee(h) \) is integral for all coroots \( \alpha^\vee \) of \( G \) (which are the roots of \( ^t G \)). This identification allows us to associate to a nilpotent orbit \( ^t \mathcal{O} \) in \( ^t \mathfrak{g} \) a dominant weight for \( G \). Namely, we can choose \( e \in ^t \mathcal{O} \) and let \( e, h, f \) span an \( \mathfrak{sl}_2 \)-subalgebra of \( ^t \mathfrak{g} \) with \( h \in ^t \mathfrak{h} \). Then by Dynkin-Kostant theory, \( h \) is well-defined up to \( W \)-conjugacy, and by \( \mathfrak{sl}_2 \)-theory, \( h \) takes integral values at the coroots of \( G \). Hence, \( h \) determines an element of
We refer to the dominant weight of $G$ thus obtained as the Dynkin weight of $^t\mathcal{O}$, and we denote by $\mathcal{D} \subset \Lambda_+$ the set of Dynkin weights of all nilpotent orbits in $^t\mathfrak{g}$.

In the previous section, we saw that $\omega_0 = 2\rho_4$ (made dominant for $GL(n)$). This is nothing more than the Dynkin weight of the nilpotent orbit $\mathcal{O}_d$ (we are identifying $G$ with $^tG$), since $\mathcal{O}_d$ intersects the Lie algebra of $L_4$ in the regular orbit. This result and a similar one for Richardson orbits in other groups (along with calculations in groups of low rank) have led a number of people to conjecture that $\mathcal{D}$ is a subset of $\Lambda^{loc}_+$ (see [5]). We wish to state a precise conjecture about how $\mathcal{D}$ sits inside of $\Lambda^{loc}_+$. To this end, we assign to each $^t\mathcal{O}$ in $^t\mathfrak{g}$ a finite cover of $\mathcal{O} = d(\mathcal{O})$, where $\mathcal{O}$ is the nilpotent orbit of $\mathfrak{g}$ dual to $^t\mathcal{O}$ under Lusztig-Spaltenstein duality. Our conjecture essentially says that if we write our putative finite cover of $\mathcal{O}$ as $G/K$, then for some $\tau$ which is trivial on $K$, $\gamma(\mathcal{O}, \tau)$ equals the Dynkin weight of $^t\mathcal{O}$.

Let $\tilde{A}(\mathcal{O})$ denote the fundamental group of $\mathcal{O}$ and let $\tilde{A}(\mathcal{O})$ denote Lusztig’s canonical quotient of $A(\mathcal{O})$ (see [9], [15]). Let $N_{o,c}$ be the set of pairs $(O, C)$ consisting of a nilpotent orbit $O \in \mathfrak{g}$ and a conjugacy class $C \subset A(\mathcal{O})$. We denote by $\mathcal{N}_{o,c}$ the set of nilpotent orbits in $^t\mathfrak{g}$. In [15] a duality map $d : N_{o,c} \to \mathcal{N}_o$ is defined which extends Lusztig-Spaltenstein duality. This map is surjective and the image of an element $(O, C) \in N_{o,c}$, denoted $d((O, C))$, depends only on the image of the conjugacy class $C$ in $\tilde{A}(\mathcal{O})$.

Now given $^t\mathcal{O} \in ^t\mathfrak{g}$, Proposition 13 of [15] exhibits an explicit element $(O, C) \in N_{o,c}$ such that $d((O, C)) = ^t\mathcal{O}$. The orbit $O$ also satisfies $O = d(\mathcal{O})$ (in particular, $O$ is special). Consider the image of $C$ in $\tilde{A}(\mathcal{O})$, which we will also denote by $C$. We suspect that this conjugacy class coincides with one that Lusztig associates to $^t\mathcal{O}$ using the special piece for $^t\mathcal{O}$ (see Remark 14 in [15]).

The canonical quotient $\tilde{A}(\mathcal{O})$ of $A(\mathcal{O})$ is always of the form $S_3, S_4, S_5$ or a product of copies of $S_2$. Hence, it is possible to describe $\tilde{A}(\mathcal{O})$ as a Coxeter group of type and then to associate to each conjugacy class $C$ in $\tilde{A}(\mathcal{O})$ a subgroup $H_C$ of $\tilde{A}(\mathcal{O})$ which is well-defined up to conjugacy in $\tilde{A}(\mathcal{O})$. Lusztig did this for the exceptional groups in [11] and we now do it for the classical groups.

First we need to describe $\tilde{A}(\mathcal{O})$ as a Coxeter group in the classical groups (where $\tilde{A}(\mathcal{O})$ is a product of copies of $S_2$). We use the description of $\tilde{A}(\mathcal{O})$ in [15]. Let $\lambda = [\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{a_k}]$ be the partition corresponding to $\mathcal{O}$ in the appropriate classical group of type $B, C,$ or $D$. Let $\mathcal{M}$ be the set of integers $m$ equal to some $\lambda_i$ such that

$$\lambda_i \text{ is odd and } \nu_i \text{ is odd in type } B_n$$

$$\lambda_i \text{ is even and } \nu_i \text{ is even in type } C_n$$

$$\lambda_i \text{ is odd and } \nu_i \text{ is even in type } D_n$$

where $\nu_i = \sum_{j=1}^i a_j$. Then from section 5 of [15], we know that the elements of $\tilde{A}(\mathcal{O})$ are indexed by subsets of $\mathcal{M}$ in type $C$ and by subsets of $\mathcal{M}$ of even cardinality in types $B$ and $D$. We choose our set of simple reflections in $\tilde{A}(\mathcal{O})$ to correspond to subsets $\{a, b\}$ of $\mathcal{M}$ with $a > b$ and where no element of $\mathcal{M}$ is both less than $a$ and greater than $b$ (in type $C$, we allow $b$ to be zero). Thus given a conjugacy class $C$ of $\tilde{A}(\mathcal{O})$ (which consists of a single element, $\omega$, since the group is abelian), we can write $\omega$ minimally as a product of simple reflections. The simple reflections used are unique, and we define $H_C$ to be the subgroup of
Definition 4.1. Two $B$-representations $V, \hat{V}$ are called $G/B$-equivalent if
\[ H^i(G/B, S^i(V^*)) \simeq H^i(G/B, S^i(\hat{V}^*)). \]
for all $i, j \geq 0$.

Our main result is for $GL(n)$, and we identify $G$ with $^4 G$.

**Theorem 4.2.** Fix the partition $d$ and let $n_{d,p} = n_{\omega_p}$. For any $p$ the $B$-representations $n_{d,p}$ are all mutually $G/B$-equivalent to each other. Thus, we have for any $p$ that

$$H^i(G/B, S^j(n_{d,p}^*)) \cong H^i(G/B, S^j(n_{d,0}^*))$$

and the latter equals $0$ if $i > 0$ and equals $\mathbb{C}^j[O_{d^*}]$ if $i = 0$ (the algebraic functions on the dual orbit of degree $2j$).

The last sentence is just a re-formulation of the Hinich, McGovern, and Panyushev results. Before proving the theorem, we prove two lemmas which rely on the following proposition.

**Proposition 4.3.** Let $\tilde{V} \subset V$ be representations of $B$ such that $V/\tilde{V} \cong \mathbb{C}_\mu$ is a one-dimensional representation of $B$ corresponding to the character $\mu$. Let $\alpha$ be a simple root, and let $P_\alpha$ be the parabolic subgroup containing $B$ and having $\alpha$ as the only positive root in its reductive part. If $V$ extends to a $P_\alpha$-representation and $\langle \alpha^\vee, \mu \rangle = -1$, then $V, \tilde{V}$ are $G/B$-equivalent.

**Proof.** Consider the exact sequence $0 \to \tilde{V} \to V \to \mathbb{C}_\mu \to 0$ and its dual $0 \to \mathbb{C}_{-\mu} \to V^* \to V^* \to 0$. By Koszul, we have $0 \to S^{j-1}(V^*) \otimes \mathbb{C}_{-\mu} \to S^j(V^*) \to S^j(V^*) \to 0$ is exact for all $j \geq 0$. By the lemma of Demazure [7], $H^*(G/B, S^{j-1}(V^*) \otimes \mathbb{C}_{-\mu}) = 0$ for all $j \geq 1$ since $V^*$ is $P_\alpha$-stable and $\langle \alpha^\vee, -\mu \rangle = 1$ (here our $B$ corresponds to the positive roots, hence the difference with Demazure’s convention). The result follows from the long exact sequence in cohomology. $\square$

We need to introduce notation to describe the $B$-stable subspaces of $n$. It is clear that if $g_\alpha$ belongs to a $B$-stable subspace $U$ of $n$, then so does $g_\beta$ for all positive roots $\beta$ with $\alpha \preceq \beta$ (where $\preceq$ denotes the usual partial order on positive roots). Hence it is enough to describe $U$ by the positive roots $\alpha$ such that $g_\alpha \subset U$ and $\alpha$ is minimal among all positive roots with this property. In this case, we say that $\alpha$ is minimal for $U$.

List the simple roots of $G$ as $\alpha_1, \ldots, \alpha_{n-1}$. Then any positive root of $G = GL(n)$ is of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, which we denote by $[i, j]$. We can express the usual partial order on the positive roots as $[i, j] \preceq [i', j']$ if and only if $i' \leq i$ and $j \leq j'$. We can then specify $U$ by its minimal positive roots, namely a collection of intervals $[i, j]$ such that for any two intervals $[i, j]$ and $[i', j']$ with $i \leq i'$, we have $j \geq j'$. We will say that $U$ is partially specified by the interval $[i, j]$ if the root $[i, j]$ is minimal for $U$ (although there may be other minimal roots). Let us also say that $U$ is $i$-stable if $U$ is stable under the action of the parabolic subgroup $P_{\alpha_i}$.

Let $U$ be a $B$-stable subspace of $n$ which is partially specified by the interval $[a, b]$ and which is either $(a-1)$-stable or $(b+1)$-stable. Let $U'$ be the subspace of $n$ which is specified by the same intervals as $U$ except that $[a, b]$ is replaced by the two intervals $[a - 1, b]$ and $[a, b + 1]$. Then $U$ and $U'$ are $G/B$-equivalent. This is simply an application of Proposition 4.3 where $\mu$ is the root $[a, b]$ and $\alpha$ is either $\alpha_{a-1}$ or $\alpha_{b+1}$. We refer to the $G/B$-equivalence of $U$ and $U'$ as the basic move for $a - 1$ (respectively, for $b + 1$). We now state two lemmas which rely solely on the basic move.

**Lemma 4.4.** Let $U$ be a subspace of $n$ which is partially specified by an interval $[a, b]$ and such that $U$ is $i$-stable for $b < i < d$ for some $d$. Let $U'$ be the subspace of $n$ specified by the
Proof of Theorem 4.2. For an orbit corresponding to $U$ the basic move to the roots $U^*$ fact that $U$ stability for $b$ result follows by induction on $d$. Let $b$. This can be seen by applying the basic move to $[b_1, b_{i+1}]$ and the interval $[b_1 + 1, b_{i+1}]$. Then $U$ is $G/B$-equivalent to $U''$. A similar statement holds even if we omit the interval $[b_1 + 1, b_{i+1}]$ or $[b_0, b_1]$ from the specification of $U$.

Proof. Let $U_1$ be specified by the same intervals as $U$, except with the above intervals replaced by the intervals

$$[b_0, b_1] \cup [b_1 + 1, b_2 - 1] \cup [b_2 + 1, b_3 - 1] \cup \cdots \cup [b_{l-1} + 1, b_l - 1] \cup [b_l + 1, b_{l+1}]$$

Then $U_1$ is seen to be $G/B$-equivalent to $U$ by applying the the basic move to the roots $[b_1 + 1, b_2 - 1], [b_2 + 1, b_3 - 1], \ldots, [b_{l-1} + 1, b_l - 1]$ since $U_1$ is stable for $b_2, b_3, \ldots, b_l$. The stability for $b_l$ follows from the assumption that no interval of $U$ is of the form $[b_1, a_1]$.

Let $U_2$ be specified by the same intervals as $U_1$, except we replace the interval $[b_0, b_1]$ with $[b_0, b_1 - 1]$ and the interval $[b_1 + 1, b_{l+1}]$ with $[b_1 + 1]$. Then $U_2$ is $G/B$-equivalent to $U_1$. This can be seen by applying the basic move to $[b_0, b_1 - 1]$ (as $U_2$ is stable for $b_1$) and applying the basic move (in reverse) to $[b_1, b_{l+1}]$ (as $U_1$ is $b_1$-stable). Here we are using the fact that $U$ contains intervals of the form $[b_0 - 1, a_2]$ and $[a_3, b_{l+1} + 1]$.

The proof is completed by observing that $U_2$ is stable for $b_1, b_2, \ldots, b_{l-1}$, so we can apply the basic move to the roots $[b_1 + 1, b_2 - 1], [b_2 + 1, b_3 - 1], \ldots, [b_{l-1} + 1, b_l - 1]$, arriving at $U''$.

Proof of Theorem 4.2. For an orbit corresponding to $d$ and a local system corresponding to $p$, we have computed $\omega_p$ in Section 2. We recall that $c$ is the g.c.d. of the parts of $d$ and $m_a$ was defined by the equation $m_a c = \sum_{i \geq 0} \mu_i a + 2i + 1$, where $\mu_i$ is the multiplicity of $i$ as a part in $d^*$. Write $k$ for one less than the largest part of $d^*$, and set $s_i = \sum_{j \geq i} m_j c$ (an empty sum is by convention equal to zero).

By Proposition 2.2, we compute that $n_{d,p} = n_{\omega_p}$ is specified by the set of intervals

$$I_i = [s_{i+1} + pm_i, s_i + pm_{i-1}]$$
where \( k \geq i \geq -k + 1 \). In the cases where an interval is not minimal or does not correspond to a root (this can happen for example when \( p = 0 \)), we disregard it from our specification of the subspace.

The difference between the left endpoint of \( I_i \) and the right endpoint of \( I_{i+2} \) will be denoted \( \Delta_i \). So

\[
\Delta_i = (c - p)m_{i+1} + pm_i.
\]

For \( i \geq 1 \) we have \( \Delta_i \leq \Delta_{i-1} \leq \Delta_{i-2} \) since \( m_{i-1} = m_i \) and \( m_i \leq m_{i-2} \) when \( i \geq 1 \).

**Step 1.** Application of Lemma 4.4 to each of the intervals \( I_i \) shows that \( n_{d,p} \) is \( G/B \)-equivalent to the subspace \( W \) of \( n \) defined by the set of intervals

\[
I_{i,j} = [s_{i+1} + pm_i - j + 1, s_{i-1} + pm_{i-2} - j]
\]

where \( 1 \leq j \leq \Delta_i \) if \( k \geq i \geq 1 \) and where \( 1 \leq j \leq \Delta_{i-2} \) if \( 0 \geq i \geq -k + 1 \).

**Step 2.** Now for each \( r \) in the range \( k \geq r \geq 1 \), starting with \( r = k \) and working down to \( r = 1 \), we will modify the intervals \( I_{r+1,j} \) and \( I_{r+1,j} \) for \( j > pm_{r+1} \) (in the cases where those intervals are defined) to obtain a new subspace of \( n \) which is \( G/B \)-equivalent to \( W \).

First, we will modify \( I_{r+1,j} \) for \( \Delta_{r+1} < j \leq \Delta_{r-1} \). Consider the intervals

\[
I_{r-1,j} \cup I_{r-3,j} \cup \cdots \cup I_{r-5,j} \cup I_{r-3,j} \cup I_{r-1,j}
\]

for \( \Delta_{r-1} < j \leq \Delta_{r-1} \) and the intervals

\[
I_{r-1,j} \cup I_{r-3,j} \cup \cdots \cup I_{r-3,j} \cup I_{r-1,j}
\]

for \( \Delta_{r+1} < j \leq \Delta_{r-1} \). These yield a situation where we can apply Lemma 4.5 a total number of \( \Delta_{r-1} - \Delta_{r-1} = p(m_{r-1} - m_{r-1}) \) times to each of the intervals \( I_{r+1,j} \) for \( \Delta_{r+1} < j \leq \Delta_{r-1} \). This will replace \( I_{r+1,j} = [s_{r+2} + pm_{r+1} - j + 1, s_r + pm_{r-1} - j] \)

with \( [s_{r+2} + pm_{r-1} - j + 1, s_r + pm_{r-1} - j] \).

Second, we will modify both \( I_{r+1,j} \) and \( I_{r+1,j} \) for \( pm_{r+1} < j \leq \Delta_{r+1} \). Consider the intervals

\[
I_{r-1,j} \cup I_{r-3,j} \cup \cdots \cup I_{r-5,j} \cup I_{r-3,j} \cup I_{r-1,j}
\]

for \( \Delta_{r+1} < j \leq (\Delta_{r-1} - \Delta_{r-1}) + \Delta_{r+1} \) and the intervals

\[
I_{r+1,j} \cup I_{r-1,j} \cup \cdots \cup I_{r-3,j} \cup I_{r-1,j}
\]

for \( pm_{r+1} < j \leq \Delta_{r+1} \). We again apply Lemma 4.5 a total number of \( \Delta_{r-1} - \Delta_{r-1} \) times by modifying the pair of intervals \( I_{r+1,j} \cup I_{r+1,j} \). Then for \( pm_{r+1} < j \leq \Delta_{r+1} \) we replace \( I_{r+1,j} \cup I_{r+1,j} \) with \( [s_{r+2} + pm_{r+1} - j + 1, s_r + pm_{r-1} - j] \cup [s_{r+2} + pm_{r-1} - j + 1, s_r + pm_{r-1} - j] \).

We do this for all \( r \) starting with \( r = k \) and working backward to \( r = 1 \) (note we haven’t done anything to the intervals \( I_{1,j} \)).

**Step 3.** At this point our subspace of \( n \) is specified by the following intervals: \([s_i - l + 1, s_{i-2} - l]\) for \( i \geq 2 \) with \( 1 \leq l \leq (c - p)m_i \) and for \( 1 \geq i \) with \( 1 \leq l \leq (c - p)m_{i-2} \), and those \( I_{i,j} \) where \( j \leq pm_i \) for \( i \geq 1 \) and \( j \leq pm_{i+2} \) for \( i \leq 0 \).

Fix \( r \geq 1 \). We may assume by induction on \( r \) that our subspace is \( G/B \)-equivalent to one partially specified by the following intervals (the case \( r = 1 \) is already true): \([s_i - l + 1, s_{i-2} - l]\) for \( 1 \leq i \leq r \) and \( 1 \leq l \leq cm_i \); and \([s_i - l + 1, s_{i-2} - l]\) for \( 0 \geq i \geq -r + 2 \) and \( 1 \leq l \leq cm_{i-2} \). We want to show that our subspace is \( G/B \)-equivalent to one partially specified by the
previous intervals together with the cases where \( i = r + 1 \) and \( i = -r + 1 \) in the above formulas.

Let \( J_{i,l} = [s_{i+1} - l + 1, s_{i-1} - l] \). Consider the intervals

\[
I_{r+1,j} \cup J_{r-2,j+(c-p)m_{r-1}} \cup \cdots \cup J_{r-r+2,j+(c-p)m_{r-1}} \cup I_{r-1,j}
\]

for \( 1 \leq j \leq pm_{r+1} \) and the intervals

\[
J_{r-2,l} \cup \cdots \cup J_{r-1,l}
\]

for \( 1 \leq l \leq (c-p)m_{r-1} \). We apply Lemma 4.5 a total number of \((c-p)(m_{r-1} - m_{r+1})\) times to the pair of intervals \( I_{r+1,j} \cup I_{r-1,j} \) for \( 1 \leq j \leq pm_{r+1} \). We thus replace \( I_{r+1,j} \cup I_{r-1,j} \) with \([s_{r+2} + pm_{r+1} - j + 1, s_{r-1} + (p-c)m_{r-1} - j]\). These intervals are just \([s_{r+1} - l + 1, s_{r-1} - l] \cup [s_{r+1} - l + 1, s_{r-1} - l] \) for \((c-p)m_{r+1} < l \leq cm_{r+1} \). Hence, by induction on \( r \) we see that our original subspace is \( G/B \)-equivalent to the subspace specified by the intervals \( J_{i,l} \) where \( 1 \leq l \leq cm_{i+1} \) for \( i \geq 1 \) and where \( 1 \leq l \leq cm_{i-1} \) for \( i \leq 0 \). This subspace is independent of \( p \) which completes the proof.

\( \square \)

Example 4.6. We illustrate the idea of the theorem for Example 2.3.

For \( p = 0 \), the initial intervals are \([3, 3] \cup [9, 9] \) (we disregard those intervals which are not minimal or which do not correspond to a root).

For \( p = 1 \), the initial intervals are \([1, 3] \cup [3, 5] \cup [5, 9] \cup [9, 10] \).

For \( p = 2 \), the initial intervals are \([2, 3] \cup [3, 7] \cup [7, 9] \cup [9, 11] \).

After step 1, for \( p = 0 \), we are already done with \([1, 6] \cup [2, 7] \cup [3, 8] \cup [7, 9] \cup [8, 10] \cup [9, 11] \).

For \( p = 1 \), we get \([1, 4] \cup [2, 7] \cup [3, 8] \cup [5, 9] \cup [8, 10] \cup [9, 11] \).

For \( p = 2 \), we get \([1, 5] \cup [2, 6] \cup [3, 8] \cup [6, 9] \cup [7, 10] \cup [9, 11] \).

For step 2, there is no effect for this example.

For step 3, for \( p = 1 \), we replace \([1, 4] \cup [5, 9] \) with \([1, 6] \cup [7, 9] \).

For \( p = 2 \), we replace \([1, 5] \cup [6, 9] \) with \([1, 6] \cup [7, 9] \) and we replace \([2, 6] \cup [7, 10] \) with \([2, 7] \cup [8, 10] \).

The result is now independent of \( p \).

Remark 4.7. An immediate consequence of the theorem is that the \( G \)-saturation of \( n_{d,p} \) is independent of \( p \) and coincides with the closure of \( O_{d_1} \). In general, one could define a map from \( \mathcal{L}_r \) to \( \mathcal{N}_n \) by first applying \( \gamma \) (for \( \mathcal{L} \)) to obtain an element \( \lambda \in \mathfrak{h} \), and then taking the \( G \)-saturation of \( \mathcal{N}_\lambda \). The referee (and we believe Ostrik too) has suggested that for \( GL(n) \) this map could be independent of the rational representation (and hence would just map to the dual orbit). Theorem 4.2 supports this conjecture in the case of a local system. In groups of other types, it is likely that there is something interesting to be said, but it is already clear from Conjecture 3.1 that the answer is going to be more complicated.

References


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