Math551 Midterm   Oct. 22, 2018

Instructions

- This is a "closed-book" Exam: do not use any book, calculator, or paper except this exam booklet.

- Organize your work in an unambiguous order. Show all necessary steps.

- **Answers given without supporting work may receive 0 credit!**

- Questions will be taken from from Chapters 1,2,3,4,5,7.

1. Consider finding a root of $f(x) = \log(x) - x + 1$ using Newton's method with initial guess $x_0 = 2$.

   (a) Show that $x = 1$ is a root of f and write down the Newton iteration.

   (b) Does the iteration converge linearly, quadratically, or some other order? If it is linear, what is the rate? Justify your answer by citing a theorem or giving direct proof.

2. Consider finding a root of $f(x) = \int_{-\infty}^{x} e^{-s^2}\,ds - x^2$ using Bisection. Note that $\int_{-\infty}^{\infty} e^{-s^2}\,ds = 2\int_{-\infty}^{0} e^{-s^2}\,ds = \sqrt{\pi}$.

   (a) There is a root $x > 0$. Bracket the root using the intermediate value theorem.

   (b) Using your interval, what is the error after n steps? How many iterations would be required to find a point within 2^{-10} of the actual root?

3. Consider the fixed point iteration defined by $x_{n+1} = g(x_n)$ where $g(x) = \log(x) + 2$.

   (a) Show that there is a fixed point for $x > 2$, for example by bracketing the root using the intermediate value theorem.

   (b) Does the iteration converge? If it does, state how quickly it converges (linearly, quadratically, or otherwise).

4. (a) Give an example of a matrix with no LU decomposition.

   (b) Find the PA $=$ LU decomposition for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 4 & 7 \end{bmatrix}$$

5. (a) Define the condition number for a matrix A.

   (b) For the exact solution x to $Ax = b$ and an approximate solution x_k, prove

   $$\frac{\|x - x_k\|}{\|x\|} \leq \kappa(A)\frac{\|b - Ax_k\|}{\|b\|}.$$

   (c) Compute the condition number for the following matrix, stating the norm used.

   $$A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$$

6. Consider solving $A\mathbf{x} = \mathbf{b}$ using Jacobi iteration where

   $$A = \begin{bmatrix} 101 & 100 \\ 99 & 100 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 201 \\ 199 \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

   (a) Find the first three terms of the iteration, x_0, x_1, x_2.

   (b) Will the iteration converge? Find the infinity norm of the iteration matrix.