Math551 Practice Final

Instructions

- The exam will be "closed-book" Exam: do not use any book, calculator, or paper except this exam booklet.
- Organize your work in an unambiguous order. Show all necessary steps.
- **Answers given without supporting work may receive 0 credit!**
- [Comments are added in brackets as hints for studying.]
- Anything on the practice midterm is also possible, so review those problems and topics.

1. Find a degree 3 interpolating polynomial through the points $(-1,-3),(0,-1),(1,-1),(3,5)$. Simplify the expression to the form $a_0 + a_1x + a_2x^2 + a_3x^3$.

   Write the expression for the interpolation error for a function passing through the points $(-1,-3),(0,-1),(1,-1),(3,5)$. Give an explicit bound at $x = 2$, assuming $|f^{(iv)}(c)| \leq 1$.

   [Know how to use both Lagrange and Newton divided differences to find a polynomial.]

2. Consider finding a least-squares fit of the planar model $z = ax + by + c$ for the data (x_i, y_i, z_i);
$$(0,0,1),(3,1,3),(1,2,0),(-1,-1,0).$$
   Set up, but do not solve the normal equations associated with the least-squares problem.

3. Approximate $\int_0^1 f(x)\,dx$ with the quadrature rule $I_h(f) = \omega_1 f(1/4) + \omega_2 f(3/4)$. Find the weights ω_1 and ω_2 such that the degree of accuracy is maximal. What is the degree of accuracy in this case?

4. Define the matrix
$$A = \begin{bmatrix} 100 & 1 & 1 & 1 \\ 1 & 50 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 2 & -50 \end{bmatrix}$$
   (a) Starting from $\mathbf{x}_0 = [1,1,1,1]^T$, compute two steps of the power iteration method (that is, $\mathbf{x}_1$ and $\mathbf{x}_2$).

   (b) Estimate the largest magnitude eigenvalue, using the Rayleigh quotient.

5. Consider a continuous, piecewise linear approximation $v(x)$ to the function $f(x) = e^x$ with two subintervals of different lengths, $t_0 = -2, t_1 = 0, t_2 = 1$. Estimate the error on each subinterval and get a single upper bound on the error $|f(x) - v(x)|$ valid on the whole interval.

6. Consider solving the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

using conjugate gradients starting from $\mathbf{x}_0 = [0, 0, 0]^T$ as initial guess.

(a) Compute a single iteration to get $\mathbf{r}_1, \mathbf{x}_1, \mathbf{p}_1$.

(b) Write an error bound for the error after three iterations. [Hint: you won't need to compute $\mathbf{x}_3$.]

7. For the 2D model problem (the Poisson equation with finite differences), with N+1 intervals in each direction, what is the condition number (use big O notation). What is the expected number of iterations needed to solve the equation with Jacobi, Gauss-Seidel, Conjugate Gradients, and Multigrid methods? (For this question only, an answer is all we are looking for.)

8. Compute the QR decomposition of the following matrix

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$