Homework 4: Due Mar. 11th, 2016

1. **Markov Matrices**  Show that all eigenvalues $\lambda$ of a Markov Matrix $A$ satisfy $|\lambda| \leq 1$. (Hint: for any eigenvector $v$ of $A^T$, we can scale so that $\max_i |v_i| = 1$. Show that $|(A^T v)_i| \leq 1$.)

2. **Convergence of Markov Matrices**  Suppose $A$ is a Markov matrix, so $\lambda_1 = 1$. Suppose that $|\lambda_i| < 1$ for $i = 2, \ldots, n$ and assume that $A$ is diagonalizable with eigenbasis $\{v_i\}_{i=1}^n$. For initial vector $w_0$, let $w_0 = \sum_{i=1}^n c_i v_i$. Find $\lim_{m \to \infty} A^m w_0$.

3. If $A$ and $B$ are Hermitian matrices, does it follow that $AB$ is a Hermitian matrix?

4. A Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is called positive definite if $\langle Av, v \rangle > 0$ for every nonzero vector $v \in \mathbb{C}^n$. Show that $A$ is positive definite if and only if it has positive eigenvalues.

5. **Rayleigh Quotient**  For a Hermitian matrix $A$, we define the the Rayleigh Quotient

$$R(A, w) = \frac{w^* A w}{w^* w}.$$ 

Let $\{v_i\}_{i=1}^n$ denote an orthonormal eigenbasis for $A$.

(a) For an eigenvector $v_i$, what is $R(A, v_i)$?

(b) If we take $w = v_1 + v_2$, what is $R(A, w)$?

(c) If we take $w = \sum_{i=1}^n \alpha_i v_i$, what is $R(A, w)$?

(d) Show that

$$\max_i \lambda_i = \max_{w \in \mathbb{C} \setminus 0} R(A, w)$$

(e) Construct an example to show that (d) is not true if $A$ is not Hermitian.