Math 235 Practice Midterm 1.

Instructions: Exam time is 2 hours. You are allowed one sheet of notes (letter-size paper, both sides). Calculators, the textbook, and additional notes are *not* allowed. *The answers are not full solutions. For the exam, justify all your answers carefully.*

Q1.

(a) Let A be a 2×2 matrix with $\det A = 3$. Compute the determinant of the matrix

$$B = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & 1 & 2 \\ & & 3 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

Answer:

$$\det B = -90$$

(b) Let C be a 3×3 matrix which is not invertible and let D be the 4×4 matrix

$$D = \begin{bmatrix} & & & 0 \\ & C & & 0 \\ & & & 0 \\ r & s & t & 1 \end{bmatrix}$$

where r, s, and t are real numbers. Explain why D is not invertible.

Answer: $\det C = 0$, since it is not invertible. Computing, we find $\det D = \det C = 0$, so D is not invertible.

Q2. Find a basis for Col A and Nul A for the following matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & -1 & -3 \\ 1 & 1 & 4 & 0 & -1 & 0 & -1 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 & -1 \end{bmatrix}$$

Answer:

A basis for Nul A is

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

A basis for Col A is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Q3. Let C be a 2×3 matrix such that $C\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^2$ and let D be a 3×2 matrix such that $D\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

a) Explain why the product DC is never invertible.

Answer: C must have a nontrivial null space, since it has at most two pivot columns. If $v \in NulC$, then $v \in NulDC$, thus DC is not invertible.

b) Is the product CD always invertible?

Answer: No, for example

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Q4. Let P_2 denote the space of all polynomials of degree less than or equal to 2.

a) Does the set $\mathcal{B} = \{(t-1)(t-2), (t+1)(t+2), t\}$ form a basis for P_2?

Answer: No. $(t-1)(t-2) - (t+1)(t+2) + 6t = 0$

b) The set $\mathcal{B} = \{t^2 + t + 1, t^2 + 2t + 1, 3t + 1\}$ forms a basis for P_2. Find the coordinates of $q(t) = 3t^2 + t - 1$ in the $\mathcal{B}$ basis.

Answer:

$$[q(t)]_{\mathcal{B}} = \begin{bmatrix} -7 \\ 10 \\ -4 \end{bmatrix}$$

Q5 a) Let V denote the vector space of all 2 by 2 matrices. Is the map $T : V \to \mathbb{R}$ given by $T(A) = \det A$ a linear transformation?

Answer: No, for example $T(2I) = 4T(I) \neq 2T(I)$

b) Let V be the set of all continuous functions f where $f(0) < f(1)$. Is V a vector space?

Answer: No, for example the function $f(x) = x$ is an element of V, but $(-1)f$ is not in V.

c) Let V be the set of all odd functions, that is $f(-x) = -f(x)$ for every real number x. Is V a vector space?

Answer: Yes, we can check the three to be a subspace.

Q6. Consider the sphere of radius 1 centered at the origin in $\mathbb{R}^3$, whose volume is $\frac{4}{3}\pi$. We transform the sphere via a linear transformation whose matrix in the standard basis is given by

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

What is the volume of the resulting shape?

Answer:

$$\frac{8\pi}{3}$$

What is $\det A^9$ for the above A?

$$\det A^9 = 512.$$