Math 235 Practice Midterm 1.

Instructions: Exam time is 2 hours. You are allowed one sheet of notes (letter-size paper, both sides). Calculators, the textbook, and additional notes are *not* allowed. Justify all your answers carefully.

Q1.

(a) Compute the reduced row echelon form of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 1 & 5 & -3 & 1 \\ -2 & -4 & 0 & -1 \\ 3 & 5 & 1 & 7 \end{bmatrix}$$

(b) Is the matrix A invertible?

(c) Which entries are pivot entries?

(d) When solving the equation $A\mathbf{x} = \mathbf{b}$ with this A, which of the following are possible: there are no solutions, there is one unique solution, there are infinitely many solutions. Justify your answer.

Q2. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$.

(a) Find the general solution of the equation $A\mathbf{x} = \mathbf{b}$. Write your solution in vector form.

(b) Using your answer to part (a) or otherwise, find the general solution of the equation $A\mathbf{x} = \mathbf{0}$.

(c) Does the equation $A\mathbf{x} = \mathbf{c}$ have a solution for every vector $\mathbf{c}$ in $\mathbb{R}^4$? Justify your answer carefully.

Q3. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

(a) Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ linearly independent?

(b) Do the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ span $\mathbb{R}^3$?

Justify your answers carefully.

Q4.

(a) Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects onto the y-axis and then rotates clockwise by $\pi/4$ radians. Find the standard matrix of S. Is S one-to-one? Is S onto?

(b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation that maps $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Write the matrix corresponding to T. Is T one-to-one? Is T onto?

Q5.

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 7 \\ -1 & 1 & 5 \end{bmatrix}$$

(b) Using your answer to part (a) or otherwise, solve the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 7x_3 & = & 3 \\ -x_1 & + & x_2 & + & 5x_3 & = & 5 \end{array}$$

Q6. Consider the equation $BC(2A - 3X)DE = F$ for an unknown $n \times n$ matrix X. Assume that A, B, C, D, E, and F are all invertible $n \times n$ matrices.

(a) Write a solution X in terms of A, B, C, D, E, and F. Is this solution unique? Explain why or why not.

(b) Can we allow any of the matrices A, B, C, D, E, or F to be singular and still guarantee that a solution X exists? Justify your answer.