ALGEBRAIC GEOMETRY: FRIDAY, MAY 12

1. Differentials and the canonical divisor

Picking up where we left off:

Definition 1.1. Let X be a nonsingular variety of dimension n over an algebraically closed field. Then, the **tangent sheaf** of X is $\mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$, and the **canonical sheaf** of X is $\omega_X = \bigwedge^n \Omega_{X/k}$. This is an invertible sheaf, so coincides with $\mathcal{L}(D)$ for some divisor D. We call this divisor the **canonical divisor** and denote it K_X.

If X is projective, we define the **geometric genus** of X to be $p_g(X) = \Gamma(X, \omega_X)$.

While we do not have time to get into many details about the sheaf of differentials and canonical divisor, they are incredibly important sheaves and divisors. They control topological properties like the genus, deformation theory, singularities, and much much more. For example, the following is true:

Theorem 1.2. *If X and Y are birational projective nonsingular varieties, then $p_g(X) = p_g(Y)$.*

Let's compute the canonical divisor for at least one variety.

Example 1.3. Let $X = \mathbb{P}^1_k$. Then, $X = U_1 \cup U_2 = \mathbb{A}^1 \cup \mathbb{A}^1$. For simplicity, let x be the coordinate on the first $\mathbb{A}^1$ and y the coordinate on the second, so on the intersection, $y = 1/x$.

Then, on U_1, Ω is the module associated to $k[x]dx$ and on U_2 it is $k[y]dy$. On the overlap, $y = 1/x$, so $dy = -1/x^2 dx$.

What sheaf is this? It is an invertible sheaf, such that the generator on one chart is $-1/x^2$ times the generator on the other. Because this has a pole of order 2, this is precisely the sheaf $\Omega_{X/k} = \omega_X = \mathcal{O}(-2)$ on $\mathbb{P}^1$. Furthermore, because there are no elements of $\Gamma(U_1, \Omega)$ that are regular (via this transition map) on U_2, there are no global sections, so $p_g = \Gamma(X, \mathcal{O}(-2)) = 0$.

2. Riemann-Roch

Now, let's combine everything we've been talking about in one very useful theorem.

Suppose X is a nonsingular projective variety over an algebraically closed field and D a divisor on X. Let $l(D)$ denote the dimension of the vector space $\Gamma(X, \mathcal{L}(D))$ (equivalently, $l(D)$ is the dimension of the linear system $|D|$, plus 1). Thus, $l(D)$ 'counts' the number of effective divisors linearly equivalent to D.

Now suppose X is a curve.

Theorem 2.1. *Riemann-Roch Let D be a divisor on a curve of genus g. Then,*

$$l(D) - l(K_X - D) = \deg D - g + 1.$$

The standard proof uses cohomology, so we will not prove this, but rather focus on many nice applications. Note that because $l(K_X - D) \geq 0$, this implies

$$l(D) \geq \deg D - g + 1.$$

First, an observation:

Lemma 2.2. *If $l(D) > 0$, then $\deg D \geq 0$. If $l(D) > 0$ and $\deg D = 0$, then $D \sim 0$.*

Proof. If $l(D) > 0$, then there exists an effective divisor D_0 linearly equivalent to D, and $\deg D = \deg D_0$, but $\deg D_0 \geq 0$ because D_0 is effective.

If $\deg D = \deg D_0 = 0$, then D is linearly equivalent to an effective divisor of degree 0, of which there is only one: $D_0 = 0$. □

Corollary 2.3. If $\deg D < 0$, then $l(nD) = 0$ for all $n > 0$, and if $\deg D = 0$, then $l(nD) \neq 0$ if and only if $nD \sim 0$.

We can use the Riemann-Roch theorem to ask what happens when $\deg D > 0$.

Corollary 2.4. If $\deg D > 0$, then for $n \gg 0$, $l(nD) = n \deg D - g + 1 > 0$.

Proof. For $n \gg 0$, $\deg K_X - nD < 0$, so $l(K_X - nD) = 0$ □

It also tells us the degree of K_X:

Corollary 2.5. The degree of the canonical divisor is $\deg K_X = 2g - 2$.

Combining the previous facts about degree 0 divisors, we have:

Corollary 2.6. The canonical divisor on a genus 1 curve is $\omega_X \cong \mathcal{O}_X$.

And, finally, it tells us:

Corollary 2.7. A divisor D on X is very ample if $\deg D \geq 2g + 1$.

Proof. First, because $\deg D \geq 2g + 1$, $\deg K_X - D \leq 2g - 2 - (2g + 1) \leq -3 < 0$ so by Riemann-Roch,

$$l(D) = \deg D - g + 1 \geq g + 2.$$

Thus, D gives us a rational map $\phi : X \to \mathbb{P}^N$, where $N \geq g + 1$. We would like to show ϕ is an embedding. To do this, we need to show two things:

(1) for any $p, q \in X$, there exists a section $s \in \Gamma(X, \mathcal{L}(D))$ such that $s(p) = 0$ but $s(q) \neq 0$. (This implies: $\phi(p) \neq \phi(q)$, so the map is injective topologically, and not all sections of $\mathcal{L}(D)$ vanish at q, so the map ϕ is defined at all points $q \in X$.)

(2) for any $p \in X$, the set $\{s \in \Gamma(X, \mathcal{L}(D)) \mid s \in m_p\mathcal{L}_p\}$ spans the vector space $m_p\mathcal{L}_p/m_p^2\mathcal{L}_p$. (This says: the sections 'separate tangent vectors', and implies that the map $\mathcal{O}_{\mathbb{P}^n} \to \phi_*\mathcal{O}_X$ is surjective–we check this on stalks, and at a point p, this is just $\mathcal{O}_{\mathbb{P}^n,p} \to \mathcal{O}_{X,p}$, but by assumption the image of the maximal ideal in $\mathcal{O}_{\mathbb{P}^n,p}$ generates m_p/m_p^2, which implies the map is surjective; see Hartshorne Lemma II.7.4 for details.)

So, let's check. We know

$$l(D) = \deg D - g + 1 \geq g + 2.$$

Suppose p and q are two points of X. Then, $\deg D - p - q = \deg D - 2 \geq 2g - 1$, so $\deg(K_X - (D - p - q)) \leq 2g - 2 - (2g - 1) = -1$, so $l(K_X - (D - p - q)) = 0$. So, by Riemann Roch,

$$l(D - p - q) = \deg D - 2 - g + 1.$$

Combining these two equations, we have $l(D - p - q) = l(D) - 2$. Similarly, computing with $D - p$ instead of $D - p - q$, one has $l(D - p) = l(D) - 1$ and $l(D - p - q) = l(D - p) - 1$.

The equality $l(D - p) = l(D) - 1$ means that there exists some global section of $\mathcal{L}(D)$ that *does not vanish* at p. Indeed, any section that vanishes at p can be written as $D' + p$, where $D' \in \mathcal{L}(D - p)$, by definition. Because $l(D - p) < l(D)$, not every section in $\mathcal{L}(D)$ is of this form, i.e. there is a section not vanishing at p. This shows the map determined by D from $X \to \mathbb{P}^N$ is in fact a morphism.

Now, suppose $p \neq q$. Because $l(D-p-q) = l(D-p) - 1$, this implies that there is a section D' of $D - p$ that *does not vanish* at q. So, there is a section of $\mathcal{L}(D)$ of the form $D' + p$, which necessarily vanishes at p, that does not vanish at q. This verifies condition (1).

Now, suppose $p = q$. Similar to the above computation, this implies that there is a section vanishing at p (i.e. in $m_p \mathcal{L}_p$) that does not vanish to order two (i.e. not in $m_p^2 \mathcal{L}_p$). So, because m_p/m_p^2 has dimension 1, the section generates $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$.

Therefore, the map induced by D is an embedding. □

Corollary 2.8. On $\mathbb{P}^1$, $g = 0$ so any divisor of positive degree is very ample.

Corollary 2.9. On any curve, if $\deg D > 0$, D is ample.

Proof. If $\deg D > 0$, then for $n \gg 0$, $\deg nD \geq 2g + 1$, so nD is very ample. □

Example 2.10. Let's use this to understand all genus 1 curves. Because $K_X = 0$, for any divisor D with $\deg D > 0$, we have $l(K - D) = l(-D) = 0$ because $\deg(-D) < 0$. So, Riemann-Roch says

$$l(D) = \deg D - g + 1 = \deg D.$$

First, this shows that a divisor of degree 1 or 2 *cannot* be very ample even though it has positive degree: $l(D) = 1$ means the map induced by D is a map to $\mathbb{P}^0$, so not an embedding, and $l(D) = 2$ means the map is to $\mathbb{P}^1$, but a genus 1 curve cannot be isomorphic to $\mathbb{P}^1$ because they have different genera, so it is also not an embedding.

However, this says if $\deg D = 3$, the map is an embedding. So, there exists a closed embedding $\phi : X \to \mathbb{P}^2$ such that $\phi^* \mathcal{O}(1) = \mathcal{L}(D)$. Because $\deg D = 3$, this implies that the hyperplane section in $\mathbb{P}^2$ restricts to be a divisor of degree 3 on X, i.e. there are 3 intersection points of a general hyperplane section and X. This implies that the image of X is defined by a cubic equation in $\mathbb{P}^2$, so all genus 1 curves can be embedded as cubics in $\mathbb{P}^2$.