ALGEBRAIC GEOMETRY: WEDNESDAY, MAY 10

1. Linear systems

Definition 1.1. An invertible sheaf $\mathcal{L}$ on a scheme X is **ample** if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer n such that for all $m \geq n$, $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated.

Exercise 1.2. If X is noetherian, $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is ample for any $m > 0$.

Ample sheaves do not have to be very ample, but it turns out that the following is true:

Proposition 1.3. *Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ an invertible sheaf. Then, $\mathcal{L}$ is ample if and only if there exists an integer m such that $\mathcal{L}^m$ is very ample with respect to $X \to \text{Spec } A$.*

To understand when divisors could be ample, we need to understand their global sections.

Definition 1.4. If $\mathcal{L}$ is an invertible sheaf on a nonsingular projective variety, for any $s \in \Gamma(X, \mathcal{L})$, we can define the **divisor of zeros** $(s)_0 \in CaCl(X)$: write $X = \cup U_i$ where $\mathcal{L}|_{U_i} \cong \mathcal{O}$, and let f_i be the image of s under this map. Let $(s)_0 = \{U_i, f_i\}$.

Proposition 1.5. *If X is a nonsingular projective variety over an algebraically closed field and D_0 a divisor on X, let $\mathcal{L} = \mathcal{L}(D_0)$ be the associated invertible sheaf. Then:*

1. *For any $s \in \Gamma(X, \mathcal{L})$, $(s)_0$ is an effective divisor linearly equivalent to D_0.*
2. *Every effective divisor linearly equivalent to D_0 is of the form $(s)_0$ for some s.*
3. *For $s, s' \in \Gamma(X, \mathcal{L})$, $(s)_0 = (s')_0$ if and only if $s = \lambda s'$ for some $\lambda \in k^*$.*

Definition 1.6. The **complete linear system** $|D_0|$ is the set of effective divisors linearly equivalent to D_0. This is in bijection with $(\Gamma(X, \mathcal{L}) - \{0\})/k^*$. The **dimension** of $|D_0|$ is its dimension as a projective variety.

If $|D_0|$ is a complete linear system on a variety X with basis of global sections $\{s_i\}$, we often denote the associated rational map to $\mathbb{P}^n$ sending $x \to [s_0(x) : \cdots : s_n(x)]$ by $|D_0| : X \dashrightarrow \mathbb{P}(|D_0|)$ because we are projectivizing the space of these divisors.

In the coming days, we will study these maps for curves.

2. Differentials and the canonical divisor

The previous section shows that a scheme is *projective* if it has a very ample invertible sheaf. How do you find very ample sheaves? Or even ample ones? For example, there exists non-projective varieties with no ample divisors.

As a first step, we need to explore what divisors exist on arbitrary varieties. There is one we always know of, called the **canonical divisor**. To define it, we begin with differentials.

Definition 2.1. Let k be a field, B a finitely generated k-algebra, and M a B-module. A k **derivation of** B **to** M is a map $d : B \to M$ such that:

1. d is additive
2. $d(bb') = bdb' + b'db$
3. $da = 0$ for all $a \in k$

The **module of differential forms** $\Omega_{B/k}$ is quotient of the free B-module generated by $\{db \mid b \in B\}$ by the relations $d(b+b') = db + db'$, $d(bb') = bdb' + b'db$, and $da = 0$. There is a derivation $d : B \to \Omega_{B/k}$ given by $b \mapsto db$.

This module satisfies the following universal property: for any B-module M and derivation $d' : B \to M$, there exists a unique factorization of d' by $B \to \Omega_{B/k} \to M$.

Example 2.2. If $B = k[x_1, \ldots, x_n]$, then $\Omega_{B/k}$ is the free B-module generated by $dx_1, \ldots, dx_n$.

Definition 2.3. If $X = \operatorname{Spec} A$ is an affine scheme over k, the sheaf of differentials on X is the sheaf of modules $\Omega_{X/k} = \widetilde{\Omega_{A/k}}$.

If X is any scheme over k, we can construct $\Omega_{X/k}$ by gluing together the sheaves on the affine charts.

Example 2.4. If $X = \mathbb{A}^n_k$, then $\Omega_{X/k}$ is the free $\mathcal{O}_X$-module generated by $dx_1, \ldots, dx_n$.

Using pure algebra (and properties of differentials) one can prove:

Theorem 2.5. *Let X be a nonsingular variety over an algebraically closed field of dimension n. Then, $\Omega_{X/k}$ is locally free of rank n.*

Definition 2.6. Let X be a nonsingular variety of dimension n over an algebraically closed field. Then, the **tangent sheaf** of X is $\mathcal{H}om(\Omega_{X/k}, \mathcal{O}_X)$, and the **canonical sheaf** of X is $\omega_X = \bigwedge^n \Omega_{X/k}$. This is an invertible sheaf, so coincides with $\mathcal{L}(D)$ for some divisor D. We call this divisor the **canonical divisor** and denote it K_X.

If X is projective, we define the **geometric genus** of X to be $p_g(X) = \Gamma(X, \omega_X)$.

While we do not have time to get into many details about these, they are incredibly important sheaves and divisors. They control topological properties like the genus, deformation theory, singularities, and much much more.