ALGEBRAIC GEOMETRY: MONDAY, MAY 8

1. Projective morphisms

Now, we will use invertible sheaves to define morphisms to projective space.

Example 1.1. On $\mathbb{P}^n$, we have the invertible sheaf $\mathcal{O}(1)$. This has n global sections: $x_0, \ldots, x_n$. (These are all global sections of the sheaf because, thinking of it as a Cartier divisor, these are the elements s for which $s \cdot 1/x_0$ is a section on all of the affine charts, so they glue together to form a global section.)

In a seemingly different direction, these global sections are also the *coordinates* on $\mathbb{P}^n$. In general, taking a line bundle $\mathcal{L}$ on an integral scheme X with global sections $\{s_0, \ldots, s_n\}$, we get a map $X \dashrightarrow \mathbb{P}^n$ by sending any point $x \in X$ to $[s_0(x) : \cdots : s_n(x)]$, where $s_i(x)$ is the evaluation of the function s_i at x. This makes sense: for integral schemes, $\mathcal{L} \subset \mathcal{K}$, where $\mathcal{K}$ is the function field of X, so the elements of $\mathcal{L}$ are rational functions on X.

Provided the sections don't simultaneously vanish, this gives a morphism $|\mathcal{L}| : X \to \mathbb{P}^n$.

The main focus of this section will be to explain the terminology associated to morphisms of line bundles.

Proposition 1.2. *Let A be a ring and let X be a scheme over A.*

(1) If $\phi : X \to \mathbb{P}^n_A$ is a morphism over A, then $\mathcal{L} = \phi^*\mathcal{O}(1)$ is an invertible sheaf on X, with global sections $s_i = \phi^*(x_i)$ that generate $\mathcal{L}$.

(2) Conversely, if $\{s_0, \ldots, s_n\}$ are global sections of an invertible sheaf $\mathcal{L}$ which generate $\mathcal{L}$ (i.e. for every point $x \in X$, the stalk $\mathcal{O}_{X,x} \cong \mathcal{L}_x$ is generated by the elements $\{s_i\}$) then there exists a unique morphism $\phi : X \to \mathbb{P}^n_A$ over A such that $\mathcal{L} \cong \phi^*\mathcal{O}(1)$ and $s_i = \phi^*(x_i)$.

Proof. For (a), $\phi^*\mathcal{O}(1)$ is invertible by definition because $\mathcal{O}(1)$ is invertible. By localizing at any point and using the definition of ϕ^*, we see that the sections s_i generate $\mathcal{L}$.

To prove (b), consider the sets $X_i = \{x \in X \mid (s_i)_x \notin m_x\mathcal{L}_x\}$ where m_x is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. These are open and cover X because the s_i generate $\mathcal{L}$. We will show that each $X_i \to D(x_i) \subset \mathbb{P}^n$, and illustrate this for $i = 0$. Write $D(x_0) = \text{Spec } A[y_1, \ldots, y_n]$ where $y_i = x_i/x_0$. Consider the ring map $A[y_1, \ldots, y_n] \to \Gamma(X_0, \mathcal{O}_{X_0})$ defined by $y_i \mapsto s_i/s_0$. (This is well defined: on X_0, any point satisfies $(s_0)_x \notin m_x\mathcal{L}_x \cong m_x$ so the quotient is an element $\mathcal{O}_{X_0}$.) Because $Hom(X_0, D(x_0)) \cong Hom(A, \mathcal{O}_{X_0})$, this gives a morphism $X_0 \to D(x_0)$. We leave it as an exercise to verify these morphisms glue to give $X \to \mathbb{P}^n$ with the required properties. □

Now, we want to study when the morphism is a closed immersion. We need $\mathcal{L}$ to have 'enough sections' to make this happen, and for the sections to generate $\mathcal{L}$ to make the morphism defined on all of X.

Definition 1.3. An invertible sheaf $\mathcal{L}$ on a scheme X over Y is **very ample** with respect to Y if there exists an immersion $i : X \to \mathbb{P}^n_Y$ such that $i^*\mathcal{O}(1) = \mathcal{L}$.

Example 1.4. On $\mathbb{P}^n_k$, $\mathcal{O}(1)$ is very ample over k by definition.

Exercise 1.5. On $\mathbb{P}^n$, the sheaf $\mathcal{O}(d)$ is the pullback of $\mathcal{O}(1)$ under the d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$. So, $\mathcal{O}(d)$ is very ample.

In general, not every 'positive degree' sheaf defines an embedding. The previous examples say that if $X = \mathbb{P}^1$ and p is a point on X, then $\mathcal{L}(p) = \mathcal{O}(1)$ is very ample. This is not true for other curves, as we will see later this week.

We need some preliminary definitions:

In general, if M is an A-module, we can construct an associated sheaf of modules on $X = \text{Spec } A$ as follows. Let

$$\tilde{M}(U) = \{s : U \to \cup_{p \in U} M_p \mid \exists V \text{ containing } p \text{ such that } s|_V = m/f \text{ where } m \in M, f \in A, f \neq 0 \text{ on } V\}.$$

(This is analogous to the construction of $\mathcal{O}_X$ from A.)

Definition 1.6. An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is **quasi-coherent** if there exists an open covering $X = \cup \text{Spec } A_i$ where $\mathcal{F}|_{A_i} = \tilde{M}_i$ for some A_i-module M_i. It is **coherent** if each M_i is a finitely generated A_i-module.

Exercise 1.7. Show that the invertible sheaves $\mathcal{L}(D)$ are coherent.

We care about quasi-coherent sheaves for many reasons. One is that we can prove things like:

Lemma 1.8. *Let $X = \text{Spec } A$ and $f \in A$. Let $U = D(f)$. Suppose $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.*

(1) If $s \in \Gamma(X, \mathcal{F})$ such that $s|_U = 0$, then $f^n s = 0$ for some $n > 0$.
(2) If $s \in \Gamma(U, \mathcal{F})$, then there exists $n > 0$ such that $f^n s \in \Gamma(X, \mathcal{F})$.
(3) If $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules and $\mathcal{F}$ is quasi-coherent, then the associated sequence of global sections is exact.

Theorem 1.9. *If X is a projective scheme over k and $\mathcal{F}$ a coherent sheaf on X, then $\Gamma(X, \mathcal{F})$ is a finite dimensional k-vector space.*

Back to morphisms:

Definition 1.10. An invertible sheaf $\mathcal{L}$ on a scheme X over Y is **very ample** with respect to Y if there exists an immersion $i : X \to \mathbb{P}^n_Y$ such that $i^*\mathcal{O}(1) = \mathcal{L}$.

Definition 1.11. A sheaf $\mathcal{F}$ on X is **globally generated** if there exists global sections $\{s_i\} \in \Gamma(X, \mathcal{F})$ such that, for every $x \in X$, $\{s_{i,x}\}$ generate the stalk $\mathcal{F}_x$.

Definition 1.12. An invertible sheaf $\mathcal{L}$ on a scheme X is **ample** if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer n such that for all $m \geq n$, $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated.

Exercise 1.13. If X is noetherian, $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is ample for any $m > 0$.

Ample sheaves do not have to be very ample, but it turns out that the following is true:

Proposition 1.14. *Let X be a scheme of finite type over a noetherian ring A and $\mathcal{L}$ an invertible sheaf. Then, $\mathcal{L}$ is ample if and only if there exists an integer m such that $\mathcal{L}^m$ is very ample with respect to $X \to \text{Spec } A$.*