ALGEBRAIC GEOMETRY: FRIDAY, MAY 5

1. Sheaves of Modules

Definition 1.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of modules** is a sheaf $\mathcal{F}$ on X such that, for each open set $U \subset X$, the group $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$ module such that the restrictions $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the module structures via the ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$.

A **morphism** of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ is a morphism $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of modules.

Definition 1.2. An $\mathcal{O}_X$-module $\mathcal{F}$ is **free** if it isomorphic to a direct sum of copies of $\mathcal{O}$. It is **locally free** is there is an open cover $X = \cup U_i$ such that $\mathcal{F}|_{U_i}$ is free for each i. The **rank** of a free sheaf is the number of copies of $\mathcal{O}$, and the **rank** of a locally free sheaf on one of the open sets U_i is the rank of $\mathcal{F}|_{U_i}$. If X is connected, the rank is the same everywhere, so it makes sense to define the rank in general.

A locally free sheaf of rank 1 is called an **invertible sheaf.**

Definition 1.3. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $f : X \to Y$ is a morphism, then **pushforward** $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module via the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{G}$ is a $f^{-1}\mathcal{O}_Y$ module, but (exercise) there is a natural morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, and the **pullback** $f^*\mathcal{G} = f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ is a $\mathcal{O}_X$ module.

2. Invertible sheaves and divisors

Exercise 2.1. If $\mathcal{L}$ is an invertible sheaf on X, define $\mathcal{L}^{-1} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Show that $\mathcal{L}^{-1}$ is also an invertible sheaf. Show that the tensor product of two invertible sheaves is an invertible sheaf. Show that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Definition 2.2. For any scheme X, the **Picard group** of X is Pic X the set of invertible sheaves on X modulo isomorphism. This is a group under $\otimes$.

We can associate to each Cartier divisor an invertible sheaf, as follows.

Definition 2.3. Let $D = \{U_i, f_i\}$ be a Cartier divisor. Let $\mathcal{L}(D)$ be the subsheaf of $\mathcal{K}$ defined by taking $\mathcal{L}(D)$ to be the $\mathcal{O}_X$-module generated by f_i^{-1} on U_i.

Exercise 2.4. Check that $\mathcal{L}(D)$ is well-defined, i.e. that f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module on $U_i \cap U_j$.

Proposition 2.5. *Let X be a scheme.*

(1) For any Cartier divisor D on X, the sheaf $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between the Cartier divisors and invertible subsheaves of $\mathcal{K}$.

(2) If D_1 and D_2 are Cartier divisors, then $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

(3) $L(D_1) \cong L(D_2)$ if and only if $D_1 \sim D_2$.

Proof. For (1), write $D = \{U_i, f_i\}$ and consider the map $\mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}$ given by $1 \mapsto f_i^{-1}$. This is an isomorphism, so $\mathcal{L}(D)$ is an invertible sheaf. Similarly, D can be recovered from $\mathcal{L}(D)$ by taking an open set where $\mathcal{O}_{U_i} \cong \mathcal{L}(D)|_{U_i}$ and letting f_i be the inverse of the generator. For any invertible subsheaf of $\mathcal{K}$, this gives a Cartier divisor D.

We leave (2) as an exercise.

For (3), by (2) it is sufficient to show that $D = D_1 - D_2$ is principal if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. But, if $D = (f)$ for $f \in \mathcal{K}^*$, then $\mathcal{L}(D)$ is globally generated by f^{-1} so the map $\mathcal{O} \to \mathcal{L}(D)$ sending 1 to f^{-1} is an isomorphism. Similarly, if $\mathcal{O} \cong \mathcal{L}(D)$, the image of the generator 1 is an element of $\mathcal{K}^*$ so D is principal, generated by its inverse. ☐

Corollary 2.6. For any scheme X, the map $CaCl(X) \to \operatorname{Pic} X$ sending a Cartier divisor D to $\mathcal{L}(D)$ is an injective homomorphism.

Proposition 2.7. *If X is integral, then this map is an isomorphism.*

Proof. We just need to show that any invertible sheaf $\mathcal{L}$ is a subsheaf of $\mathcal{K}$, and then the result follows from our previous proposition.

Because X is integral, $\mathcal{K}$ is the constant sheaf with values $K(X)$. Consider the sheaf $\mathcal{L} \otimes \mathcal{K}$. On any open set U where $\mathcal{L}|_U \cong \mathcal{O}$, this is just isomorphic to $\mathcal{K}$. This says that X has an open cover such that the restriction of $\mathcal{L} \otimes \mathcal{K}$ to each open set is the constant sheaf $\mathcal{K}$. Because X is irreducible, this implies that $\mathcal{L} \otimes \mathcal{K}$ must in fact be a constant sheaf (exercise!) so $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$. Therefore, the injective map $\mathcal{L} \to \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$ shows that $\mathcal{L}$ is a subsheaf of $\mathcal{K}$. ☐

Corollary 2.8. On a noetherian integral separated locally factorial scheme, we have

$$\operatorname{Cl} X \cong CaCl(X) \cong \operatorname{Pic} X.$$

Example 2.9. On $\mathbb{P}^n_k$, we know $\operatorname{Cl} X \cong \mathbb{Z}$ generated by the hyperplane $x_0 = 0$, so $\operatorname{Pic} X = \mathbb{Z}$ generated by the sheaf $\mathcal{L}(x_0)$. This sheaf is denoted $\mathcal{O}(1)$.

3. Projective morphisms

Now, we will use invertible sheaves to define morphisms to projective space.

Example 3.1. On $\mathbb{P}^n$, we have the invertible sheaf $\mathcal{O}(1)$. This has n global sections: $x_0, \dots, x_n$. (These are all global sections of the sheaf because, thinking of it as a Cartier divisor, these are the elements s for which $s \cdot 1/x_0$ is a section on all of the affine charts, so they glue together to form a global section.)

In a seemingly different direction, these global sections are also the *coordinates* on $\mathbb{P}^n$. In general, taking a line bundle $\mathcal{L}$ on an integral scheme X with global sections $\{s_0, \dots, s_n\}$, we get a map $X \dashrightarrow \mathbb{P}^n$ by sending any point $x \in X$ to $[s_0(x) : \cdots : s_n(x)]$, where $s_i(x)$ is the evaluation of the function s_i at x. This makes sense: for integral schemes, $\mathcal{L} \subset \mathcal{K}$, where $\mathcal{K}$ is the function field of X, so the elements of $\mathcal{L}$ are rational functions on X.

Provided the sections don't simultaneously vanish, this gives a morphism $|\mathcal{L}| : X \to \mathbb{P}^n$.

The main focus of this section will be to explain the terminology associated to morphisms of line bundles.

Proposition 3.2. *Let A be a ring and let X be a scheme over A.*

(1) If $\phi : X \to \mathbb{P}^n_A$ is a morphism over A, then $\mathcal{L} = \phi^*\mathcal{O}(1)$ is an invertible sheaf on X, with global sections $s_i = \phi^*(x_i)$ that generate $\mathcal{L}$.

(2) Conversely, if $\{s_0, \dots, s_n\}$ are global sections of an invertible sheaf $\mathcal{L}$ which generate $\mathcal{L}$ (i.e. for every point $x \in X$, the stalk $\mathcal{O}_{X,x} \cong \mathcal{L}_x$ is generated by the elements $\{s_i\}$) then there exists a unique morphism $\phi : X \to \mathbb{P}^n_A$ over A such that $\mathcal{L} \cong \phi^*\mathcal{O}(1)$ and $s_i = \phi^*(x_i)$.