ALGEBRAIC GEOMETRY: WEDNESDAY, MAY 3

1. Cartier Divisors

So far, we have defined Weil divisors (linear combinations of closed subschemes of codimension 1) and Cartier divisors (global sections of $\mathcal{K}^*/\mathcal{O}^*$). Although these are defined quite differently, in nice cases, the Weil divisors and the Cartier divisors coincide.

Proposition 1.1. *Let X be an integral separated noetherian scheme that is locally factorial, meaning that all local rings $\mathcal{O}_{X,x}$ are unique factorization domains. Then, the group of Weil divisors* $\text{Div } X$ *is isomorphic to the group of Cartier divisors $\mathcal{K}^*/\mathcal{O}^*(X)$. Furthermore, this respects the principal divisors, so gives an isomorphism* $\text{Cl } X$ *to $CaCl(X)$.*

Example 1.2. Any regular local ring is a UFD, so this statement applies for regular ('nonsingular') noetherian separated schemes.

Corollary 1.3. By the proof of the previous proposition, we see that for any *normal* scheme the Cartier divisors are isomorphic to the subgroup of locally principal Weil divisors, as claimed at the beginning of the section.

So, on normal schemes (where Weil divisors can be defined), the Cartier divisors are a subset of the Weil divisors. If our scheme is not regular or not locally factorial, they do not have to be the same.

Example 1.4. For the quadric cone $X = \text{Spec } k[x, y, z]/(xy - z^2)$, the divisor $Y = V(y, z)$ is not (locally) principal, so is not Cartier. The class group is $\mathbb{Z}/2\mathbb{Z}$ generated by Y, so in this case, the group $CaCl(X) = 0$.

The last 'type' of divisor we will consider is an *invertible sheaf.* To define invertible sheaf, we first have to define sheaves of modules.

2. Sheaves of Modules

Definition 2.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A **sheaf of modules** is a sheaf $\mathcal{F}$ on X such that, for each open set $U \subset X$, the group $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$ module such that the restrictions $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the module structures via the ring homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$.

A **morphism** of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ is a morphism $\mathcal{F} \to \mathcal{G}$ such that $\mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of modules.

Exercise 2.2. The kernel, cokernel, and image of a morphism of $\mathcal{O}_X$-modules is a $\mathcal{O}_X$-module. The quotient sheaf of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. The sheaf of homomorphisms $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module.

Definition 2.3. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, then the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$ is called **the tensor product** and is denoted $\mathcal{F} \otimes \mathcal{G}$. It is also an $\mathcal{O}_X$-module.

Definition 2.4. An $\mathcal{O}_X$-module $\mathcal{F}$ is **free** if it isomorphic to a direct sum of copies of $\mathcal{O}$. It is **locally free** is there is an open cover $X = \cup U_i$ such that $\mathcal{F}|_{U_i}$ is free for each i. The **rank** of a free sheaf is the number of copies of $\mathcal{O}$, and the **rank** of a locally free sheaf on one of the open sets U_i is the rank of $\mathcal{F}|_{U_i}$. If X is connected, the rank is the same everywhere, so it makes sense to define the rank in general.

A locally free sheaf of rank 1 is called an **invertible sheaf.**

The terminology 'invertible sheaf' has meaning: it will mean the sheaf is invertible in the group of $\mathcal{O}_X$ modules with group operation $\otimes$.

3. Invertible sheaves and divisors

Exercise 3.1. If $\mathcal{L}$ is an invertible sheaf on X, define $\mathcal{L}^{-1} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Show that $\mathcal{L}^{-1}$ is also an invertible sheaf. Show that the tensor product of two invertible sheaves is an invertible sheaf. Show that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

With this exercise in hand, we can define the Picard group:

Definition 3.2. For any scheme X, the **Picard group** of X is Pic X the set of invertible sheaves on X modulo isomorphism. This is a group under $\otimes$.

We can associate to each Cartier divisor an invertible sheaf, as follows.

Definition 3.3. Let $D = \{U_i, f_i\}$ be a Cartier divisor. Let $\mathcal{L}(D)$ be the subsheaf of $\mathcal{K}$ defined by taking $\mathcal{L}(D)$ to be the $\mathcal{O}_X$-module generated by f_i^{-1} on U_i.

Exercise 3.4. Check that $\mathcal{L}(D)$ is well-defined, i.e. that f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module on $U_i \cap U_j$.

Proposition 3.5. *Let X be a scheme.*

(1) For any Cartier divisor D on X, the sheaf $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between the Cartier divisors and invertible subsheaves of $\mathcal{K}$.

(2) If D_1 and D_2 are Cartier divisors, then $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$.

(3) $L(D_1) \cong L(D_2)$ if and only if $D_1 \sim D_2$.

Proof. For (1), write $D = \{U_i, f_i\}$ and consider the map $\mathcal{O}_{U_i} \to \mathcal{L}(D)|_{U_i}$ given by $1 \mapsto f_i^{-1}$. This is an isomorphism, so $\mathcal{L}(D)$ is an invertible sheaf. Similarly, D can be recovered from $\mathcal{L}(D)$ by taking an open set where $\mathcal{O}_{U_i} \cong \mathcal{L}(D)|_{U_i}$ and letting f_i be the inverse of the generator. For any invertible subsheaf of $\mathcal{K}$, this gives a Cartier divisor D.

We leave (2) as an exercise.

For (3), by (2) it is sufficient to show that $D = D_1 - D_2$ is principal if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. But, if $D = (f)$ for $f \in \mathcal{K}^*$, then $\mathcal{L}(D)$ is globally generated by f^{-1} so the map $\mathcal{O} \to \mathcal{L}(D)$ sending 1 to f^{-1} is an isomorphism. Similarly, if $\mathcal{O} \cong \mathcal{L}(D)$, the image of the generator 1 is an element of $\mathcal{K}^*$ so D is principal, generated by its inverse. □

Corollary 3.6. For any scheme X, the map $CaCl(X) \to \text{Pic } X$ sending a Cartier divisor D to $\mathcal{L}(D)$ is an injective homomorphism.

Proposition 3.7. *If X is integral, then this map is an isomorphism.*

We will prove this next time!

Corollary 3.8. On a noetherian integral separated locally factorial scheme, we have

$$\text{Cl } X \cong CaCl(X) \cong \text{Pic } X.$$

Example 3.9. On $\mathbb{P}^n_k$, we know Cl $X \cong \mathbb{Z}$ generated by the hyperplane $x_0 = 0$, so Pic $X = \mathbb{Z}$ generated by the sheaf $\mathcal{L}(x_0)$. This sheaf is denoted $\mathcal{O}(1)$.