ALGEBRAIC GEOMETRY: MONDAY, MAY 1

1. Cartier Divisors

Inspired by the previous notion of divisor, we will define *Cartier divisors*, which are divisors that locally look like principal divisors (f). For example, on $\mathbb{A}^n$, every divisor is principal, so the Cartier divisors are the same as the Weil divisors. On $\mathbb{P}^n$, Weil divisors cannot be globally written as principal divsors, but locally they can: we have a cover $\mathbb{P}^n = \cup \mathbb{A}^n$, and taking any divisor D, it has a cover by $D \cap \mathbb{A}^n$, and on each of these charts, it is principal.

However, we should already expect that not all Weil divisors look like Cartier divisors: we already saw that on the quadric cone Spec $k[x, y, z]/(xy - z^2)$, the divisor $Y = V(y, z)$ is not locally principal because it cannot be written as the vanishing of one element near the singular point of the cone.

We begin with the definition of Cartier divisor, which we can make for any scheme.

Definition 1.1. Let X be a scheme. For any $U =$ Spec A an affine open set, let $K(U)$ be the localization of A at the multiplicatively closed subset of nonzero divisors. (For example, if A is a domain, $K(U)$ is just the fraction field.)

For any open set U, we have a similar construction: let $S(U)$ be the set of elements of $\mathcal{O}_X(U)$ which are not zero divisors in any of the local rings $\mathcal{O}_{X,x}$ for $x \in U$. Let $K(U) = S(U)^{-1}\mathcal{O}_X(U)$.

The sheafification $\mathcal{K}$ of the presheaf $U \mapsto K(U)$ is called the **sheaf of total quotient rings of** X.

We denote $\mathcal{K}^*$ the sheaf of invertible elements in $\mathcal{K}$ and $\mathcal{O}^*$ the sheaf of invertible elements in $\mathcal{O}$.

Remark 1.2. This sheaf $\mathcal{K}$ is just the replacement of the function field of an integral scheme (c.f. Exercise 3.6.)

Definition 1.3. A **Cartier divisor** on a scheme X is a global section of $\mathcal{K}^*/\mathcal{O}^*$.

More concretely, a Cartier divisor on X is described by an open cover $\{U_i, f_i\}$ where $X = \cup U_i$ and $f_i \in \mathcal{K}^*(U_i)$ such that for each i, j, $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$.

A Cartier divisor is **principal** if it is in the image of the map $\mathcal{K}^*(X) \to \mathcal{K}^*/\mathcal{O}^*(X)$, and two Cartier divisors are **linearly equivalent** if their difference is principal.

Write $CaCl(X)$ for the group of Cartier divisors modulo linear equivalence.

While this definition is abstract, think of it as follows: a Cartier divisor is a collection of principal divisors (f_i) on open sets U_i that 'glue together' to form a divisor. What should that mean? It should mean that the divisors (f_i) and (f_j) agree on the intersection $U_i \cap U_j$, i.e. have the same zeros and poles. If f_i and f_j have the same set of zeros and poles, then f_i/f_j is an invertible regular function because it has no zeros and no poles. This is precisely what the $f_i/f_j \in \mathcal{O}^*$ condition means.

Example 1.4. Let $X = \mathbb{P}^1_{x,y}$. Then, $K(X) = k(x/y)$ and $\mathcal{K}$ is the constant sheaf whose value on connected open sets are $\mathcal{K}(U) = k(x/y)$.

Let $Y = V(x)$ be the point $[0 : 1]$. This is not principal because it is not an element of $k(x/y)$, but it is locally principal: write $X = U_1 \cup U_2$ where $U_1 = D(y)$ and $U_2 = D(x)$. Let $f_1 = x/y$ and let $f_2 = 1$. Then, on $U_1 \cap U_2 = D(xy)$, $f_1/f_2 = x/y$ and $f_2/f_1 = y/x$ are both regular (because x and y are both nonzero). Also, on $D(y)$, $(f_1) = Y$: the function f_1 has a zero at

$Y = V(x)$ and has no poles because $y \neq 0$. On $D(x)$, $(f_2) = 0$ because f_2 has no zeros or poles. So, the divisor Y is locally principal and a Cartier divisor.

Similarly, $Z = V(y)$ is a Cartier divisor, and the difference $Y - Z = (x/y)$ is principal, so Y and Z are linearly equivalent.

Exercise: show any point $Z = V(ax + by)$, $a, b \in k$ not both zero, is linearly equivalent to Y.

In nice cases, the Weil divisors and the Cartier divisors coincide.

Proposition 1.5. *Let X be an integral separated noetherian scheme that is locally factorial, meaning that all local rings $\mathcal{O}_{X,x}$ are unique factorization domains. Then, the group of Weil divisors* $\mathrm{Div}X$ *is isomorphic to the group of Cartier divisors $\mathcal{K}^*/\mathcal{O}^*(X)$. Furthermore, this respects the principal divisors, so gives an isomorphism* $\mathrm{Cl}X$ *to $CaCl(X)$.*

Proof. Since UFDs are integrally closed, X satisfies the assumptions made to define Weil divisors. Furthermore, since X is integral, the sheaf $\mathcal{K}$ is the constant sheaf whose values on connected open sets are $K = \mathcal{O}_{X,p}$ where p is the generic point of X (locally, the (0) ideal).

Let D_C be a Cartier divisor given by an open cover $\{U_i, f_i\}$ where $f_i \in K^*$. We can define an associated Weil divisor D_W: given any prime divisor Y, let i be some index such that $U_i \cap Y \neq \emptyset$ and let $n_Y = v_Y(f_i)$. Note this is well-defined: if j is another index such that $U_j \cap Y$ is nonempty, then $f_i/f_j, f_j/f_i \in \mathcal{O}^*$ and hence f_i/f_j is invertible, so $v_Y(f_i/f_j) = 0$. Therefore, $v_Y(f_i) = v_Y(f_j)$. Then, define $D_W = \sum n_Y Y$.

Conversely, let D_W be any Weil divisor. For any $x \in X$, consider the induced divisor D_x on Spec $\mathcal{O}_{X,x}$ (only the parts of D_W that have non-zero restriction to this subscheme). Because $\mathcal{O}_{X,x}$ is a UFD, ClSpec $\mathcal{O}_{X,x} = 0$, so D_x is principal and $D_x = (f_x)$ for some $f_x \in K^*$. Furthermore, (f_x) differs only from D_W by finitely many prime divisors (those parts of (f_x) or D_W that do not pass through x), so there is an open set U_x such that D_W coincides with (f_x) on U_x. Covering X by finitely many of these sets, we obtain that D_W corresponds to the Cartier divisor $D_C = \{U_x, f_x\}$. (Note that if f, f' give the same Weil divisor on any open set, then $f/f' \in \mathcal{O}^*$: this uses that X is normal and is just saying that, if two divisors have the same zeros and poles, then they differ by a unit.)

Finally, we leave as an exercise to verify that these constructions are inverse to each other and send principal divisors to principal divisors, so we obtain the claimed isomorphisms. ☐

Example 1.6. Any regular local ring is a UFD, so this statement applies for regular (= smooth over $\mathbb{C}$) noetherian separated scheme.