ALGEBRAIC GEOMETRY: FRIDAY, APRIL 28

1. Divisors

In what follows, we assume all schemes are noetherian integral separated schemes that are regular in codimension 1.

Definition 1.1. Div$X$ is the free abelian group generated by the prime divisors (codimension 1 irreducible closed sets) of X.

If $f \in K(X)^*$ is any rational function on X, then the **divisor of** f is the divisor in Div$X$

$$(f) = \sum v_Y(f)Y.$$

Any divisor that equals (f) for some $f \in K^*$ is called a **principal divisor.**

The **class group** of X, denoted $\mathrm{Cl}(X)$, is the group Div$X$ modulo the subgroup of principal divisors. If D_1, D_2 are elements of Div$X$, we write $D_1 \sim D_2$ if they have the same image in $\mathrm{Cl}(X)$.

Last time, we looked at the class groups of $\mathbb{A}^n$ and $\mathbb{P}^n$:

Proposition 1.2. *Let A be a noetherian integral domain. Then, A is a UFD if and only if $X = \mathrm{Spec}\, A$ is normal and $\mathrm{Cl}(X) = 0$.*

Corollary 1.3. Because $A = k[x_1, \ldots, x_n]$ is a UFD, $\mathbb{A}^n = \mathrm{Spec}\, A$ has $\mathrm{Cl}(X) = 0$.

Proposition 1.4. *Let $X = \mathbb{P}^n_k$. For any $D \in \mathrm{Div}(X)$, $D = \sum n_i Y_i$ where each Y_i is a hypersurface of degree d_i, and we define $\deg D := \sum n_i d_i$. Let H be the hyperplane $(x_0 = 0)$. Then:*

1. *For any $f \in K^*$, $\deg(f) = 0$.*
2. *For any $D \in \mathrm{Div}(X)$, if $\deg D = d$, then $D \sim dH$.*
3. *The function $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is an isomorphism.*

Now, we will use these to understand class groups of other varieties.

Proposition 1.5. *Let X satisfy the assumption at the beginning of the section, $Z \subset X$ a proper closed subset, and $U = X - Z$. Then,*

1. *There is a surjective homomorphism $\mathrm{Cl}X \to \mathrm{Cl}U$ given by $D = \sum n_i Y_i \mapsto \sum n_i(Y_i \cap U)$ (ignoring any $Y_i \cap U$ that is empty)*
2. *If $\mathrm{codim}_X Z \geq 2$, then $\mathrm{Cl}X \to \mathrm{Cl}U$ is an isomorphism.*
3. *If Z is irreducible of codimension 1, then there is an exact sequence*
$$\mathbb{Z} \to \mathrm{Cl}X \to \mathrm{Cl}U \to 0$$
where the first map sends 1 to $1 \cdot Z$.

Proof. For (1), the map on Div is well defined because any prime divisor Y on X restricts to either a prime divisor on U or $U \cap Y$ is empty. For $f \in K^*$, $(f) = \sum n_i Y_i$, then f is also a rational function on U and $(f)_U = \sum n_i(Y_i \cap U)$ so the image of a principal divisor is principal. Therefore, the map on ClX is well-defined. Also, every prime divisor on U is the restriction of its closure in X, so the map is surjective.

For (2), this follows because Div and Cl were defined by generic points of codimension 1 subsets, so removing subsets of codimension 2 does not change the definitions.

Finally, for (3), we consider $\ker \mathrm{Cl}X \to \mathrm{Cl}U$. This only consists of divisors whose support is contained in Z, so if Z is irreducible, the kernel is just generated by $1 \cdot Z$. □

We can use this proposition to understand the class group of several other varieties:

Example 1.6. Let $X = \mathbb{P}^2$ and Z be an irreducible plane curve of degree d. Then, from the previous two propositions, $\mathrm{Cl}(X - Z) = \mathbb{Z}/d\mathbb{Z}$.

Example 1.7. Let $X = \mathrm{Spec}\ A$ where $A = k[x, y, z]/(xy - z^2)$. Graphing this (or using some previous exercises), X is a cone in $\mathbb{A}^3$. Specifically, X is the cone over the conic $xy - z^2$ in $\mathbb{P}^2$.

We will show that $\mathrm{Cl}X = \mathbb{Z}/2\mathbb{Z}$. Let Y be a ruling of the cone, $Y = V(y, z)$. Then, Y is a prime divisor, so by the previous proposition, there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}X \to \mathrm{Cl}X - Y \to 0.$$

Furthermore, we can understand $\mathrm{Cl}(X - Y)$: notice that, set theoretically, Y is the closed subset $V(y)$ (because $y = 0$ implies that $z^2 = 0$, so the points of $V(y)$ and Y are the same. Also, $(y) = 2 \cdot Y$: at the generic point p of Y (which is all we use to define the divisor), the local ring is $\mathcal{O}_{X,p} = k[x, y, z]/(xy - z^2)_{(y,z)}$. So, we invert everything outside of (y, z), and the maximal ideal consists of (the image of) functions in the ideal (y, z). Here, the maximal ideal m is generated by just z, i.e. $m = (z)$ because x is invertible, so $y \in (z)$. Therefore, $(y) = 2 \cdot Y$ as $y = 0$ implies $z^2 = 0$, and z^2 vanishes to order 2 in the maximal ideal. This says that $2 \cdot Y$ is principal, so $2 \cdot Y \sim 0 \in \mathrm{Cl}X$.

Also, because $X - Y$ is also the complement of (y) in X, we have $X - Y = X - V(y)$, and $X - V(y) = D(y) = \mathrm{Spec}\ A_y$. Therefore,

$$X - Y = \mathrm{Spec}\ k[x, y, z]/(xy - z^2)_y = \mathrm{Spec}\ k[x, y, y^{-1}, z]/(xy - z^2).$$

Because y is invertible in this ring, we can eliminate x and conclude

$$X - Y = \mathrm{Spec}\ k[y, y^{-1}, z],$$

which is a UFD, so $\mathrm{Cl}(X - Y) = 0$.

So, to complete the proof that $\mathrm{Cl}X = \mathbb{Z}/2\mathbb{Z}$, it suffices to show that $Y \not\sim 0 \in \mathrm{Cl}X$. Because X is normal (exercise!) it suffices to show that (y, z) is not a principal ideal in A. (This uses the algebraic proof of the UFD proposition: roughly, A is a UFD if and only if every prime ideal of height 1 is principal, and if this prime ideal is not principal, then it is a nontrivial element in $\mathrm{Cl}X$.) We leave this as an algebra exercise.

What this example is hinting at is that *singular* varieties have interesting (local) class groups. We will come back to this.

We can prove several other results in this direction.

Example 1.8. If X satisfies our main assumption, then so does $X \times \mathbb{A}^1$, and $\mathrm{Cl}X \cong \mathrm{Cl}(X \times \mathbb{A}^1)$.

Example 1.9. Let $X = V(xy - zw) \subset \mathbb{P}^3$. Then, $\mathrm{Cl}X \cong \mathbb{Z} \times \mathbb{Z}$. The generators of the class group are the two rulings of the surface. Try this as an exercise, using the exact sequence above with respect to the closed subset Z that is one of the rulings and the previous example that $\mathrm{Cl}Y \cong \mathrm{Cl}(Y \times \mathbb{A}^1)$ (here: the complement of one ruling should be a ruling $\times \mathbb{A}^1$...)

The key in the previous example/exercise is to define a map $\pi^* : \mathrm{Cl}\mathbb{P}^1 \to \mathrm{Cl}\mathbb{P}^1 \times \mathbb{P}^1$ by 'pullback': consider either projection $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, and for any divisor $D = \sum n_i Y_i$ on $\mathbb{P}^1$, let $\pi^* D = \sum n_i \pi^{-1} Y_i$. Then, show that $\mathrm{Cl}\mathbb{P}^1 \times \mathbb{P}^1$ is $\pi_1^* \mathrm{Cl}\mathbb{P}^1 \oplus \pi_2^* \mathrm{Cl}\mathbb{P}^1 = \mathbb{Z} \oplus \mathbb{Z}$. This allows us to define a 'multidegree' of any divisor on $\mathbb{P}^1 \times \mathbb{P}^1$: we think of divisors as (a, b), $a, b \in \mathbb{Z}$, where a is the degree of the divisor pulled back from the first $\mathbb{P}^1$ and b the degree from the second.