ALGEBRAIC GEOMETRY: WEDNESDAY, APRIL 26

1. Divisors

For this section, we will assume that our schemes are *regular in codimension 1*, i.e if $x \in X$ is the generic point of a codimension 1 subscheme, then $\mathcal{O}_x$ is regular. If X is a nonsingular variety, all local rings $\mathcal{O}_x$ are regular, so X satisfies this condition. More generally, if X is a noetherian normal scheme, then $\mathcal{O}_x$ is an integrally closed domain of dimension 1, which is regular.

So, in what follows, we assume all schemes are noetherian integral separated schemes that are regular in codimension 1.

Definition 1.1. A **prime divisor** on a scheme X is an integral closed subscheme of codimension 1. We define $\mathrm{Div}X$ to be the free abelian group generated by all prime divisors.

A **Weil divisor** is an element $D = \sum n_i Y_i \in \mathrm{Div}X$. By definition, each Y_i is a prime divisor and each n_i is an integer such that only finitely many n_i are nonzero.

If $n_i \geq 0$ for all i, then we say D is **effective.**

If Y is any prime divisor, let y be its generic point, so the ring $\mathcal{O}_{X,y}$ is a *discrete valuation ring* with associated valuation v_Y and quotient field K. For any $f \in K^*$ (which defines a nonzero rational function on X), $v_Y(f) \in \mathbb{Z}$, and if $v_Y(f) = n > 0$, then we say f has a **zero** of order n along Y, and if $v_Y(f) = -n < 0$, we say f has a **pole** of order n along Y.

Definition 1.2. If $f \in K^*$ is any rational function on X, then the **divisor of** f is the divisor in $\mathrm{Div}X$

$$(f) = \sum v_Y(f) Y.$$

Any divisor that equals (f) for some $f \in K^*$ is called a **principal divisor.**

Definition 1.3. The **class group** of X, denoted $\mathrm{Cl}(X)$, is the group $\mathrm{Div}X$ modulo the subgroup of principal divisors. If D_1, D_2 are elements of $\mathrm{Div}X$, we write $D_1 \sim D_2$ if they have the same image in $\mathrm{Cl}(X)$.

Exercise 1.4. Prove that, on $\mathbb{A}^1$, every element of $\mathrm{Div}X$ is principal. (Possible hint: write down the rational function that gives you the divisor?)

More generally, we have the following proposition. (The proof is purely algebraic, so we omit it.)

Proposition 1.5. *Let A be a noetherian integral domain. Then, A is a UFD if and only if $X = \mathrm{Spec}\, A$ is normal and $\mathrm{Cl}(X) = 0$.*

Corollary 1.6. Let $X = \mathbb{A}^n$. Then, $X = \mathrm{Spec}\, k[x_1, \ldots, x_n]$ which is a UFD, so $\mathrm{Cl}(X) = 0$.

Let's briefly think about $X = \mathbb{P}^1$. What is $\mathrm{Cl}(X)$? First, what are the regular functions? We know, if X has coordinates x, y, the rational functions on X are just $k(x/y)$, because this is the quotient field of $\mathcal{O}_{X,p}$ for any point $p \in X$. But, certainly $x = 0$ is a closed subscheme of X! It turns out that there is *no* principal divisor that has this image in $\mathrm{Div}X$! The problem is essentially that, if you try $f = x/y$, then $(f) = (x = 0) - (y = 0)$. So, we will see that $\mathrm{Cl}(X)$ is not zero.

Proposition 1.7. *Let $X = \mathbb{P}^n_k$. For any $D \in \mathrm{Div}(X)$, $D = \sum n_i Y_i$ where each Y_i is a hypersurface of degree d_i, and we define $\deg D := \sum n_i d_i$. Let H be the hyperplane $(x_0 = 0)$. Then:*

(1) For any $f \in K^*$, $\deg(f) = 0$.
(2) For any $D \in \mathrm{Div}(X)$, if $\deg D = d$, then $D \sim dH$.
(3) The function $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is an isomorphism.

Proof. Let $S = k[x_0, \ldots, x_n]$ be the coordinate ring of X. If $g \in S_d$ is homogeneous of degree d, then we can factor g as $g = g_1^{n_1} \ldots g_r^{n_r}$ where $Y_i = V(g_i)$ is an integral closed subscheme of codimension 1. We define the divisor of g to be $(g) = \sum n_i Y_i$. If $\deg g_i = d_i$, then $\deg Y_i = d_i$, so $\deg(g) = \sum n_i d_i = d$. In other words, this definition of degree of divisor coincides with the degree of the defining polynomial.

If $f \in K^*$, then $f = g/h$ where g and h are homogeneous functions of the same degree, and $(f) = (g) - (h)$, so $\deg(f) = \deg(g) - \deg(h) = 0$ because f and g have the same degree.

If $D \in \mathrm{Div} X$ has degree d, then we may write $D = D_1 - D_2$ where D_1 and D_2 are effective divisors of degrees d_1 and d_2 such that $d_1 - d_2 = d$. Any effective divisor can be written (g) for some homogeneous $g \in S$ because irreducible hypersurfaces in $\mathbb{P}^n$ correspond to vanishing of principal ideals, and any effective divisor is a formal combination of these (so is a product of powers of $g_i \in S$). So, if $D_1 = (g_1)$ and $D_2 = (g_2)$, then $f = g_1/x_0^d g_2$ is homogeneous of degree 0 and

$$(f) = (g_1) - (g_2) - d(x_0) = D_1 - D_2 - dH = D - dH.$$

Therefore, $D - dH$ is principal so $D \sim dH$.

Finally, because any $D \in \mathrm{Div} X$ is equivalent to dH in $\mathrm{Cl} X$ and $\deg H = 1$, we have $\mathrm{Cl}(X) \cong \mathbb{Z}$. □