ALGEBRAIC GEOMETRY: MONDAY, APRIL 24

1. Morphisms

The conclusion of the section on separated/proper/projective morphisms:

Exercise 1.1. Show that the image of the functor $t : \mathcal{V}ar_k \to \mathcal{S}ch_k$ taking a variety V to the set of closed subsets $t(V)$ is exactly the set of integral quasi-projective schemes over k.

Hint: use the affine description of t–if V is an affine variety with coordinate ring A, then $t(V) = \text{Spec } A$–to show that for any variety V, $t(V)$ is an integral quasi-projective scheme. Then, show that, if Y is any integral (quasi-)projective scheme over k, so $Y \subset \mathbb{P}^n_k$, the set V of closed points of Y is a quasi-projective variety (a closed set of an open subvariety of the variety $\mathbb{P}^n_k$). *Comment from class: this is using k algebraically closed!*

Motivated by the previous exercise, we finally define a *variety* in the category of schemes.

Definition 1.2. A **variety** is an integral separated scheme of finite type over k. A **complete variety** is a proper integral scheme over k.

2. Divisors

Next, we move on to studying *divisors*, which are codimension one subvarieties (or subschemes), which capture much of the intrinsic geometry of the variety!

If you are following along in Hartshorne, we are (temporarily) skipping Section 5. We will come back to it, but will introduce divisors (Section 6) first.

For this section, we will assume that our schemes are *regular in codimension 1*, i.e if $x \in X$ is the generic point of a codimension 1 subscheme, then $\mathcal{O}_x$ is regular. If X is a nonsingular variety, all local rings $\mathcal{O}_x$ are regular, so X satisfies this condition. More generally, if X is a noetherian normal scheme, then $\mathcal{O}_x$ is an integrally closed domain of dimension 1, which is regular.

So, in what follows, we assume all schemes are noetherian integral separated schemes that are regular in codimension 1.

Definition 2.1. A **prime divisor** on a scheme X is an integral closed subscheme of codimension 1. We define $\text{Div}X$ to be the free abelian group generated by all prime divisors.

A **Weil divisor** is an element $D = \sum n_i Y_i \in \text{Div}X$. By definition, each Y_i is a prime divisor and each n_i is an integer such that only finitely many n_i are nonzero.

If $n_i \geq 0$ for all i, then we say D is **effective.**

If Y is any prime divisor, let y be its generic point, so the ring $\mathcal{O}_{X,y}$ is a *discrete valuation ring* with associated valuation v_Y and quotient field K. For any $f \in K^*$ (which defines a nonzero rational function on X), $v_Y(f) \in \mathbb{Z}$, and if $v_Y(f) = n > 0$, then we say f has a **zero** of order n along Y, and if $v_Y(f) = -n < 0$, we say f has a **pole** of order n along Y.

This is best motivated with the case of curves.

Example 2.2. Suppose $X = \mathbb{A}^1 = \text{Spec } \mathbb{C}[t]$. Then, the codimension 1 points of X are just the closed points $a \in \mathbb{C}$, which correspond to the maximal ideals $(t - a) \subset \mathbb{C}[t]$. Suppose Y is the origin (the point $a = 0$, or maximal ideal (t)) so the local ring $\mathcal{O}_{X,y} = k[t]_{(t)}$. The quotient field is $\mathbb{C}(t)$ and the associated valuation just counts the order of vanishing of t. To be precise: let $f \in \mathbb{C}(t)$ be any function and write $f = p(t)/q(t)$ where p, q are polynomials. Let n be the maximal integer such that t^n divides f, meaning $f/t^n = p'(t)/q'(t)$ and t does not divide $q'(t)$ (in other words, $p'(t)/q'(t) \in k[t]_{(t)}$). Then, the valuation is defined to be $v_Y(f) = n$.

We'll use some specific functions to make this clear: if $f = t^3/(3-2t)$, then $v_Y(f) = 3$. If $g = t^2(2-4t)/t^3$, then $v_Y(g) = -1$.

The terminology *zeros* and *poles* is used because that's literally what the valuation is counting in this case.

In the case above, $f = t^3/(3-2t) = t^3(1/(3-2t)) =: tf'$. The function f' is in $k[t]_{(t)}$, so is regular near the point $t = 0$, and it makes sense to say that f vanishes to order 3 near $t = 0$. Similarly, $g = (1/t)g'$ where $g' \in k[t]_{(t)}$, so g' is regular near $t = 0$, but g has a simple pole at $t = 0$.

Lemma 2.3. *If $f \in K^*$ is any nonzero rational function on X, then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset where f is regular and $Z = X - U$. Because X is noetherian, finitely many prime divisors Y are contained in Z, so it suffices to prove that there are only finitely many prime divisors in U with $v_Y(f) \neq 0$. Because f is regular along U, $v_Y(f) \geq 0$, and $v_Y(f) > 0$ if and only if Y is contained in the closed subset $V(I)$, where $I = (f) \subset A$. Because $f \neq 0$, this is a proper closed subset, so contains only finitely many irreducible subsets of codimension 1. $\square$

With the lemma in hand, it makes sense to define a generalization of the earlier example:

Definition 2.4. If $f \in K^*$ is any rational function on X, then the **divisor of** f is the divisor in $\operatorname{Div} X$

$$(f) = \sum v_Y(f) Y.$$

Any divisor that equals (f) for some $f \in K^*$ is called a **principal divisor.**

By properties of valuations, $(f/g) = (f) - (g)$, so this is often something we can readily compute.

Example 2.5. Let $f = t^3/(3-2t) \in K^*$ where $X = \mathbb{A}^1$. Then, $(f) = 3(t = 0) - (t = 3/2)$.
Let $g = t^2(2-4t)/t^3 \in K^*$. Then, $(g) = (t = 1/2) - (t = 0)$.

Exercise 2.6. The function $K^* \to \operatorname{Div} X$ given by $f \mapsto (f)$ is a group homomorphism. By definition, the image is the set of principal divisors.

Definition 2.7. The divisor **class group** of X, denoted $\operatorname{Cl}(X)$, is the group $\operatorname{Div} X$ modulo the subgroup of principal divisors.