ALGEBRAIC GEOMETRY: FRIDAY, APRIL 21

1. Projective Morphisms

Definition 1.1. Let Y be a scheme. We define **projective n-space** over Y to be the scheme $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\text{Spec } Z} Y$. By definition of the fiber product, there is a natural projection map $\mathbb{P}^n_Y \to Y$.

Definition 1.2. A morphism $f : X \to Y$ is **projective** if there exists some n such that f factors as $X \to \mathbb{P}^n_Y \to Y$, where the first map is a closed immersion and the second map is the natural projection. A morphism $f : X \to Y$ is **quasi-projective** if it factors into an open immersion $X \to X'$ followed by a projective morphism $X' \to Y$.

Example 1.3. Let k be a field and S a graded ring with $S_0 = k$ and such that S is finitely generated as a k-algebra by S_1. (Example: $S = k[x_0, \dots, x_n]$.) Then, the natural map $\text{Proj } S \to \text{Spec } k$ is projective: by hypothesis, $S = k[x_0, \dots, x_n]/I$ for some ideal I, so if $S' = k[x_0, \dots, x_n]$, there is a surjective homomorphism $S' \to S$, so a closed immersion $\text{Proj } S \to \text{Proj } S' = \mathbb{P}^n_k$, which the morphism $\text{Proj } S \to \text{Spec } k$ factors through.

In particular, $\mathbb{P}^n_k$ is projective over $\text{Spec } k$.

Finally, we conclude with:

Theorem 1.4. *A projective morphism of noetherian schemes is proper. A quasi-projective morphism of noetherian schemes is of finite type and separated.*

Proof. By the corollary of the valuative criterion, it is sufficient to prove that $\mathbb{P}^n_{\mathbb{Z}} \to \text{Spec } \mathbb{Z}$ is proper (any projective morphism factors as a closed immersion and a base change of this morphism, which will be proper if morphism is).

First, note that $\mathbb{P}^n_{\mathbb{Z}} = \cup D_+(x_i)$, where $D_+(x_i) = \text{Spec } \mathbb{Z}[x_0/x_i, \dots, x_n/x_i]$, and each of the restricted morphisms $\text{Spec } \mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \to \text{Spec } \mathbb{Z}$ is finite type, so $\mathbb{P}^n_{\mathbb{Z}} \to \text{Spec } \mathbb{Z}$ is of finite type. Now, we use the valuative criterion. Suppose we have a diagram

$$\begin{array}{ccc} U = \text{Spec } K & \longrightarrow & X = \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow \\ T = \text{Spec } R & \longrightarrow & Y = \text{Spec } \mathbb{Z}. \end{array}$$

Let ξ_1 be the image of the unique point of U in X. By induction, we may assume that ξ_1 is not contained in any of the hyperplanes $V(x_i) \subset \mathbb{P}^n_{\mathbb{Z}}$, because each $V(x_i) \cong \mathbb{P}^{n-1}$, so we assume $\xi_1 \in \cap D_+(x_i)$ and hence all x_i/x_j are regular functions at ξ_1, so $x_i/x_j \in \mathcal{O}_{\xi_1}$ for all i, j.

The morphism $U \to X$ corresponds to an inclusion of fields $k(\xi_1) \subset K$. Let $f_{ij} \in K$ be the image of x_i/x_j under this inclusion. Because R is a valuation ring, there is an associated (totally ordered) abelian group G and a valuation $v : K \to G$ with $R = \{x \in K \mid v(x) \geq 0\}$. Let $g_i = v(f_{i0})$ for each i, and choose j such that $g_j = \min\{g_i \mid 0 \leq i \leq n\}$. Then, $g_i - g_j = v(f_{ij}) \geq 0$ for each i, so $f_{ij} \in R$ for each i. Therefore, there is a homomorphism $\mathbb{Z}[x_0/x_j, \dots, x_n/x_j] \to R$ given by $x_i/x_j \to f_{ij}$ (compatible with $k(\xi_1) \to K$ by definition), so there exists a morphism $T \to D_+(x_j)$, which composed with the inclusion $D_+(x_j) \to \mathbb{P}^n_{\mathbb{Z}}$ gives the required morphism $T \to \mathbb{P}^n_{\mathbb{Z}}$.

To complete the proof, we leave as an exercise to show this morphism is unique. □

Example 1.5. As a consequence, the previous example $\mathbb{P}_k^n \to \operatorname{Spec} k$ (or more generally $\operatorname{Proj} S \to \operatorname{Spec} k$ where S is a graded ring finitely generated over k in degree 1) is a proper morphism.