ALGEBRAIC GEOMETRY: WEDNESDAY, APRIL 19

1. Proper Morphisms

Last time, we proved the valuative criterion of separatedness:

Proposition 1.1. *Let $f : X \to Y$ be a morphism of schemes and suppose X is noetherian. Then, f is separated if and only if for all valuation rings R with fraction field K and natural inclusion $U = \text{Spec } K \to T = \text{Spec } R$ with morphisms $U \to X$ and $T \to Y$ such that*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

commutes, there is at most one morphism $T \to X$ making the following diagram commute

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

and we also proved several of the following corollaries:

Corollary 1.2. Assume all schemes are noetherian in the following statements.

(1) Open and closed immersions are separated.
(2) A composition of separated morphisms is separated.
(3) Separated morphisms are stable under base extension (i.e. if $f : X \to Y$ is separated and $Z \to Y$ is another map, then the induced map $X \times_Y Z \to Z$ is separated).
(4) If $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of schemes over a base S, then the product $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated.
(5) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is separated, then f is separated.
(6) A morphism $f : X \to Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for all i.

Now, we move into proper morphisms.

Definition 1.3. A morphism $f : X \to Y$ is **universally closed** if it is closed and for any $Z \to Y$, the morphism $X \times_Y Z \to Z$ is closed.

Definition 1.4. A morphism $f : X \to Y$ is **proper** if f is separated, finite type, and universally closed.

Proper in this setting is the analogue of the usual definition of proper (the preimage of any compact set is compact) in the Zariski topology.

Example 1.5. The morphism $\mathbb{A}^1_k \to \text{Spec } k$ is not proper because it is not universally closed. For example, if we base change along $\mathbb{A}^1 \to k$, we consider $\mathbb{A}^1 \times_k \mathbb{A}^1 \to \mathbb{A}^1$, but

$$\mathbb{A}^1 \times_k \mathbb{A}^1 = \text{Spec } k[t] \otimes_k k[t] = \text{Spec } k[t_1, t_2] = \mathbb{A}^2,$$

so this is the map $\mathbb{A}^2 \to \mathbb{A}^1$. This is not closed (the image of the closed set $V(t_1 t_2 - 1)$ is the complement of the origin in $\mathbb{A}^1$, which is not closed).

Intuitively, properness should be saying 'limits exist' (whatever limits mean in algebraic geometry). (More formally: we might want to say something like 'if t_0 is a specialization of t_1, and t_1 is a point in X, then the specialization t_0 should also be a point in X.') In the context of the valuative criterion for separatedness, if a map is proper, then because t_0 is a 'limit' of t_1, being proper should say that the map $T \to X$ exists, because the image of t_1 exists in X. This is exactly the valuative criterion for properness.

Proposition 1.6. *Let $f : X \to Y$ be a morphism of finite type and suppose X is noetherian. Then, f is proper if and only if for all valuation rings R with fraction field K and natural inclusion $U = \text{Spec } K \to T = \text{Spec } R$ with morphisms $U \to X$ and $T \to Y$ such that*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

commutes, there exists a unique morphism $T \to X$ making the following diagram commute

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

The proof is similar in spirit to Friday's proof of the valuative criterion of separatedness. If you are interested in seeing the details, we can prove it next time.

Given the valuative criterion, we can similarly use it to prove many morphisms are proper.

Corollary 1.7. Assume all schemes are noetherian in the following statements.

(1) A closed immersion is proper.
(2) A composition of proper morphisms is proper.
(3) Proper morphisms are stable under base extension (i.e. if $f : X \to Y$ is proper and $Z \to Y$ is another map, then the induced map $X \times_Y Z \to Z$ is proper).
(4) If $f : X \to Y$ and $f' : X' \to Y'$ are proper morphisms of schemes over a base S, then the product $f \times f' : X \times_S X' \to Y \times_S Y'$ is proper.
(5) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is proper and g is separated, then f is proper.
(6) A morphism $f : X \to Y$ is proper if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is proper for all i.

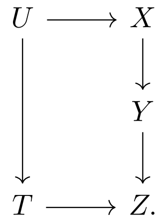
Let's prove some of these.

Proof. For (1), suppose $f : X \to Y$ is a closed immersion and we have a diagram

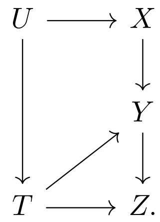
$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

Because $f : X \to Y$ is a closed immersion, $f(X)$ is a closed subscheme of Y which contains the image of the generic point of T, and because closed implies stable under specialization, it also contains the image of T. Because $X \to f(X)$ is an isomorphism, the map $T \to Y$ factors through X, as desired.

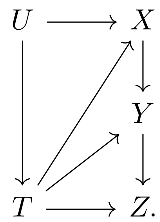
Let's also prove (2). First, as an exercise: prove that a composition of finite type morphisms is finite type. Now, composing the morphisms, we must consider a diagram



Because $Y \to Z$ is proper, there is a unique morphism $T \to Y$ making the diagram below commute:



and because $X \to Y$ is proper, there is a unique morphism making the diagram below commute:



Therefore, by the valuative criterion, the composition is proper. $\square$

Finally, we define projective morphisms. Recall that projective space over any ring A was defined as $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$. *Comment: in the definition of* Proj *, we had a graded ring* S*. The grading on* $A[x_0, \dots, x_n]$ *is obtained by setting* $\deg a = 0$ *for any* $a \in A$*, and* $\deg x_i = 1$ *for any* i*.*

Exercise 1.8. If $A \to B$ is any ring homomorphism, then $\mathbb{P}^n_B = \mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B$. In particular, for any ring A, there is a canonical homomorphism $\mathbb{Z} \to A$, so $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A$.

The exercise motivates the following definition:

Definition 1.9. Let Y be a scheme. We define **projective** n**-space** over Y to be the scheme $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$. By definition of the fiber product, there is a natural projection map $\mathbb{P}^n_Y \to Y$.

Definition 1.10. A morphism $f : X \to Y$ is **projective** if there exists some n such that f factors as $X \to \mathbb{P}^n_Y \to Y$, where the first map is a closed immersion and the second map is the natural projection. A morphism $f : X \to Y$ is **quasi-projective** if it factors into an open immersion $X \to X'$ followed by a projective morphism $X' \to Y$.

Example 1.11. If $Y = \operatorname{Spec} k$ and X is a noetherian scheme, the map $f : X \to Y$ being projective says that $X \hookrightarrow \mathbb{P}^n_k$, i.e. X is a closed subscheme of the usual projective space. We'll formalize this next time.