## ALGEBRAIC GEOMETRY: FRIDAY, APRIL 14

### 1. Separated and Proper Morphisms

We will start with the proof of the valuative criterion.

**Exercise 1.1.** Prove the following:

If $X$ is an integral scheme, then there exists a generic point $\xi$ on $X$, and $\mathcal{O}_{X,\xi}$ is a field. This is called the *function field* of $X$. If $X = \text{Spec } A$, then $\mathcal{O}_{X,\xi} \cong \text{Frac}(A)$. There is a natural map $A \to \text{Frac}(A)$ which corresponds to an inclusion of schemes $\text{Spec Frac}A \to \text{Spec } A$. Because $\text{Frac}(A) = (A - \{0\})^{-1}A$, there is only one prime ideal $\text{Frac}A$ and hence only one point of $\text{Spec Frac}A$, and its image in $\text{Spec } A$ is the generic point $\{(0)\}$.

**Proposition 1.2.** *Let $f : X \to Y$ be a morphism of schemes and suppose $X$ is noetherian. Then, $f$ is separated if and only if for all valuation rings $R$ with fraction field $K$ and natural inclusion $U = \text{Spec } K \to T = \text{Spec } R$ with morphisms $U \to X$ and $T \to Y$ such that*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

*commutes, there is at most one morphism $T \to X$ making the following diagram commute*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$

The proof will be divided into three steps.

**Step 1.**

**Lemma 1.3.** *Let $R$ be a valuation ring of a field $K$. Let $T = \text{Spec } R$ and $U = \text{Spec } K$. A morphism from $U$ to a scheme $X$ is equivalent to a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subset K$. A morphism from $T$ to a scheme $X$ is equivalent to two points $x_0, x_1 \in X$ with $x_0 \in \overline{\{x_1\}}$ and an inclusion $k(x_1) \subset K$ such that $R$ dominates $\mathcal{O}_{\overline{\{x_1\}},x_0}$, where $\overline{\{x_1\}}$ is taken to have the reduced induced structure as a subscheme of $X$.*

**Remark 1.4.** As a reminder, if $R$ and $S$ are local rings, we says $R$ dominates $S$ if $R$ contains $S$ and the maximal ideal of $R$ contains the maximal ideal of $S$.

*Proof.* Because $U$ is a one-point scheme, a morphism $f : U \to X$ is a point $x_1 \in X$ (the image of $U$) and a local homomorphism from $\mathcal{O}_X \to f_*\mathcal{O}_U$. By definition of $f_*\mathcal{O}_U$, this is necessarily the zero map at any point outside $x_1$, so we just need a local homomorphism $\mathcal{O}_{X,x_1} \to K$ (because $K$ is the structure sheaf of $\mathcal{O}_U$). By definition of $k(x_1)$, this is a nonzero map $k(x_1) \to K$ between fields, so an inclusion $k(x_1) \subset K$.

For the second statement, let $t_0$ be the closed point of $T$ and $t_1$ be the generic point. If $f : T \to X$ is a morphism, let $x_0 = f(t_0)$ and $x_1 = f(t_1)$. Since $T$ is reduced, the image factors through $Z = \overline{\{x\}}$ (exercise!), and because $Z$ is reduced and irreducible, its function field $\mathcal{O}_{Z,x_1} = k(x_1)$, so the natural map on stalks of the structure sheaf gives an inclusion $k(x_1) \subset K$. On the closed point, we have the local homomorphism $\mathcal{O}_{Z,x_0} \to R$ compatible with

$k(x_1) \subset K$ (induced by taking fraction fields) so $R$ dominates $\mathcal{O}_{Z,x_0}$. On the other hand, given $x_0, x_1$ and $k(x_1) \subset K$ such that $R$ dominates $\mathcal{O}$, the inclusion $\mathcal{O} \to R$ gives a map on Specs $T \to \text{Spec } \mathcal{O}$, and composing the natural map $\text{Spec } \mathcal{O} \to X$ from including $x_0$ into $X$, we get a map $T \to X$. □

**Step 2.**

We used the following notion in the previous step, and will use it again below.

**Definition 1.5.** A point $x_0 \in X$ is a **specialization** of $x_1 \in X$ if $x_0 \in \overline{\{x_1\}}$.

**Lemma 1.6.** *If $f : X \to Y$ is a quasi-compact morphism of schemes, then $f(X) \subset Y$ is closed if and only if it is stable under specialization.*

*Proof.* If $f(X)$ is closed, then for any point $y_1 \in f(X)$, $\overline{\{y_1\}} \subset f(X)$, so any specialization of $y_1$ is also in $f(X)$. This proves one implication.

Now, suppose $f(X)$ is closed under specialization. Because closedness depends only on the reduced structure, we may assume both $X$ and $Y$ are reduced. Also, to prove $f(X)$ is closed, it suffices to show $f(X) = \overline{f(X)}$, so we can replace $Y$ with $\overline{f(X)}$ (with reduced induced structure). In other words, assume $\overline{f(X)} = Y$. Now, let $y \in \overline{f(X)}$. We must show $y \in f(X)$. By considering a neighborhood of $y \in Y$, we may assume that $Y$ is affine, and therefore (because $f$ is quasi-compact), $f^{-1}(Y) = \cup_{i=1}^n X_i$ where each $X_i$ is affine. Because $y \in Y$ and the union is finite, $y \in \overline{f(X_i)}$ for some $i$. Let $Y_i = \overline{f(X_i)}$ with reduced induced structure, which is a closed subset of the affine scheme $Y$, so is also affine (and reduced by assumption).

Summarizing the set-up, we have $X_i = \text{Spec } A$, $Y_i = \text{Spec } B$ with $B$ reduced, such that $y \in Y_i$ and $X_i \to Y_i$ is dominant. This is equivalent to an injective ring map $B \to A$ and prime ideal $p \subset B$ (corresponding to the point $y$). By Zorn's lemma, minimal prime ideals exist, so there exists a minimal prime ideal $p'$ of $B$ contained in $p$, which corresponds to a point $y'$ such that $y$ is a specialization of $y'$. If we can show that $y' \in f(X_i)$, then because it is stable under specialization, that will imply $y \in f(X_i)$, as desired.

So, to conclude the proof, consider the injective map $B \to A$. We can localize these at the prime ideal $p'$ (localizing $A$ at $p'$ as a $B$-module) giving the injective map $B_{p'} \to A \otimes_B B_{p'}$, and $B_{p'}$ is a field because $p'$ was minimal. So, if $q'_0$ is any prime ideal in $A \otimes B_{p'}$, then $q'_0 \cap B_{p'} = (0)$. Let $q'$ be the inverse image of $q'_0$ in $A$ (under the localization map). Then, $q' \cap B = p'$, so the image of the point $x' \in X_i$ corresponding to $q'$ is $y'$, i.e. $y' = f(x')$ so $y' \in f(X_i)$. □

**Step 3.**

To prove the theorem, we will use the previous two lemmas.

*Proof.* First suppose that $f : X \to Y$ is separated and we have two morphisms $h, h' : T \to X$ making the diagram commute. This gives a morphism $h'' = (h, h') : T \to X \times_Y X$, and the restrictions of $h$ and $h'$ to $U$ are the same, so the generic point $t_1 \in T$ has $h''(t_1) \in \Delta(X)$. Because $\Delta(X)$ is closed, this implies $h''(t_0) \in \Delta(X)$, so $h$ and $h'$ send the points $t_0, t_1$ to the same points $x_0, x_1 \in X$. Because the inclusions $k(x_1) \subset K$ coming from $h, h'$ are the same, by Step 1, the maps $h$ and $h'$ are equal.

Now, suppose there is at most one map completing the diagram. To show $f$ is separated, we must show that $\Delta(X)$ is closed. By assumption, $X$ is noetherian, so $\Delta$ is quasi-compact (exercise!) so it suffices to show, by Step 2, that $\Delta(X)$ is stable under specialization. So, assume $z_1 \in \Delta(X)$ is a point and $z_0$ is a specialization. Let $K = k(z_1)$ and $\mathcal{O}$ be the local ring of $z_0$ on the closure of $z_1$ with its reduced induced structure.

*Key algebra fact* (which we won't prove): because $\mathcal{O}$ is a local ring contained in $K$, there is a valuation ring $R$ dominating $\mathcal{O}$. Then, by Step 1, we have a morphism $T = \text{Spec } R \to X \times_Y X$, and composing with the projections to $X$, we get two morphisms $T \to X$ that agree over $Y$ and agree when restricted to $U$. By assumption, these must be the same, which implies $T \to X \times_Y X$ factors through $\Delta(X)$ and $z_0 \in \Delta(X)$. Therefore, $X$ is separated. □

The point of the valuative criterion is that it is much easier to check separatedness of morphisms using this! We can prove statements like:

**Corollary 1.7.** Assume all schemes are noetherian in the following statements.

(1) Open and closed immersions are separated.
(2) A composition of separated morphisms is separated.
(3) Separated morphisms are stable under base extension (i.e. if $f : X \to Y$ is separated and $Z \to Y$ is another map, then the induced map $X \times_Y Z \to Z$ is separated).
(4) If $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms of schemes over a base $S$, then the product $f \times f' : X \times_S X' \to Y \times_S Y'$ is separated.
(5) If $f : X \to Y$ and $g : Y \to Z$ are morphisms such that $g \circ f$ is separated, then $f$ is separated.
(6) A morphism $f : X \to Y$ is separated if and only if $Y$ can be covered by open subsets $V_i$ such that $f^{-1}(V_i) \to V_i$ is separated for all $i$.

*Proof.* We will prove (1) and (3) as examples of why this 'immediately' follows from the valuative criterion. For (1), let $X \to Y$ be an open immersion (the image of $X$ is an open subscheme of $Y$). Consider any diagram as in the valuative criterion:

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

Let $g : T \to Y$ be the given map. Then, if $g(T) \subset f(X)$, there exists a unique map $T \to X$ making the diagram commute. Otherwise, there does not exist such a map. Therefore, by the valuative criterion, open immersions are separated.

Let's also prove (3). Suppose $f$ is separated and $Z \to Y$ is any other morphism, and consider the diagram:

$$\begin{array}{ccccc} U & \longrightarrow & X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & Z & \longrightarrow & Y \end{array}$$

If there exist two morphisms $T \to X \times_Y Z$ making the diagram commute, composing with the map $X \times_Y Z \to X$ along the top of the diagram, we have two morphisms $T \to X$, which must be equal because $f$ is separated. So, separated morphisms are stable under base change. □