ALGEBRAIC GEOMETRY: WEDNESDAY, APRIL 12

1. Separated and Proper Morphisms

By now, we know that the Zariski topology is not Hausdorff, but we will introduce a condition called *separatedness* capturing a similar idea. We also may want compactness properties of our schemes, beyond just compactness of the topological space, to which we will introduce *properness.*

Definition 1.1. Let $f : X \to Y$ be a morphism of schemes. The **diagonal** is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with each projection is the identity.

Example 1.2. Suppose $X = \text{Spec } A$ and $Y = \text{Spec } B$ are affine schemes. Then, the diagonal is the map $\text{Spec } A \to \text{Spec } A \otimes_B A$ that gives the identity on the projections, and we know exactly what this map is. In terms of rings, the projections are $\text{Spec } A \otimes_B A \to \text{Spec } A$ given by $A \to A \otimes_B A$ where $a \mapsto a \otimes 1$ or $1 \otimes a$. Therefore, the diagonal is the map $A \otimes_B A \to A$ that induces the identity when composed with the projection, so it is given by $a \otimes a' = aa'$.

This morphism is called the diagonal because it gives the usual diagonal map on topological spaces.

Example 1.3. Let $X = \text{Spec } k[x] = \mathbb{A}^1_k \to Y = \text{Spec } k$. The diagonal is the map

$$X = \text{Spec } k[x] \to X \times_k X = \text{Spec } k[x] \otimes_k k[x] \cong \text{Spec } k[x_1, x_2] = \mathbb{A}^2_k$$

coming from the ring map

$$\begin{aligned} \phi : k[x_1, x_2] &\to k[x] \\ x_1 &\mapsto x \\ x_2 &\mapsto x. \end{aligned}$$

What is the image of any point in X? If $a \in \mathbb{A}^1$ is a closed point, then a corresponds to a maximal ideal $(x - a)$, and the image of this ideal is $\phi^{-1}(x - a) = (x_1 - a, x_2 - a)$, which is the closed point $(a, a) \in \mathbb{A}^2$. So, the image of the diagonal is the usual diagonal $x_1 = x_2$ in $\mathbb{A}^2$.

Definition 1.4. We say that f is **separated**, or sometimes X is **separated** over Y, if the diagonal is a closed immersion. We say that that X is **separated** if it is separated over $\text{Spec } \mathbb{Z}$.

Example 1.5. The previous calculation shows that $\mathbb{A}^1$ is separated over k because the image of the diagonal is the closed subscheme $V(x_1 - x_2) \in \text{Spec } k[x_1, x_2]$.

Example 1.6. The affine line with doubled origin X is not separated over $\text{Spec } k$. The fiber product $X \times_k X$ is the affine plane with double axes and *four* origins. If p_1, p_2 are the two origins in X, then (p_1, p_1), (p_1, p_2), (p_2, p_1) and (p_2, p_2) are all in X. Only two of the origins are in the image of Δ: (p_1, p_1) and (p_2, p_2). However, the closure of the image of Δ includes all four origins, so the diagonal is not a closed immersion.

Most morphisms we've talked about so far are separated:

Proposition 1.7. *If $f : \text{Spec } A \to \text{Spec } B$ is a morphism of affine schemes, then f is separated.*

Proof. The diagonal comes from the ring map $A \otimes_B A \to A$ given by $a \otimes a' \mapsto aa'$. This is a surjective homomorphism of rings, so is a closed immersion (exercise: prove a surjection of rings gives a closed immersion on Specs). □

Corollary 1.8. A morphism $f : X \to Y$ is separated if and only if the image of Δ is a closed subset of $X \times_Y X$.

Proof. If f is separated then the image is closed by definition. Conversely, assume $\Delta(X)$ is a closed subset. We need to show Δ is a closed immersion, i.e. $\Delta(X)$ is homeomorphic to X and the map $\mathcal{O}_{X\times_Y X} \to \Delta_*\mathcal{O}_X$ is surjective. The homeomorphism is automatic because Δ composed with either projection $X \times_Y X \to X$ is the identity on X, and the surjectivity can be checked locally on affine open sets. Indeed, let $p \in X$ be any point and $U = \text{Spec } A \subset X$ a sufficiently small affine neighborhood of p such that $f(U) \subset V = \text{Spec } B \subset Y$. Then, $\Delta(p)$ is contained in the affine open set $U \times_V U = \text{Spec } (A \otimes_B A) \subset X \times_Y X$ and the previous proposition says $\mathcal{O}_{U\times_V U} \to \Delta|_{U*}\mathcal{O}_U$ is surjective, but this is just the restriction of $\mathcal{O}_{X\times_Y X} \to \Delta_*\mathcal{O}_X$. This shows $\mathcal{O}_{X\times_Y X} \to \Delta_*\mathcal{O}_X$ is surjective in a neighborhood of every point, so it is surjective. □

Separatedness is capturing a Hausdorff-like property, which can be described as follows. Let R be a DVR, i.e. a one-dimensional noetherian integrally closed local ring, like $R = k[x]_{(x)}$. These rings have two prime ideals, (0) and the unique maximal ideal m. By definition, m is a closed point and (0) is dense. Geometrically, these are *germs* of curves: imagine zooming in infinitely near to a point p on a one-dimensional variety X. Then, the structure sheaf is $\mathcal{O}_{X,p}$ is a DVR, and the associated geometric object Spec $\mathcal{O}_{X,p}$ is the germ of p on X, with maximal ideal m corresponding to the closed point and (0) the generic point (like a small neighborhood of p).

Consider our example of a non-separated scheme, the affine line with doubled origin, and maps from Spec $R - \{m\}$ (so, maps from just the generic point of R). We could map $(0) \in \text{Spec } R$ to the generic point of the non-separated scheme, but then we have two choices to extend the map to the closed point, because it could map to either of the origins. It turns out that this phenomenon of having more than one possible map from a valuation ring is equivalent to closed immersion definition of separatedness. We have to work with arbitrary valuation rings instead of DVRs, but the idea is the same.

First, an exercise so the notation makes sense:

Exercise 1.9. Prove the following:

If X is an integral scheme, then there exists a generic point ξ on X, and $\mathcal{O}_{X,\xi}$ is a field. This is called the *function field* of X. If $X = \text{Spec } A$, then $\mathcal{O}_{X,\xi} \cong \text{Frac}(A)$. There is a natural map $A \to \text{Frac}(A)$ which corresponds to an inclusion of schemes Spec Frac$A \to$ Spec A. Because $\text{Frac}(A) = (A - \{0\})^{-1}A$, there is only one prime ideal Frac$A$ and hence only one point of Spec Frac$A$, and its image in Spec A is the generic point $\{(0)\}$.

Proposition 1.10. *Let $f : X \to Y$ be a morphism of schemes and suppose X is noetherian. Then, f is separated if and only if for all valuation rings R with fraction field K and natural inclusion $U = \text{Spec } K \to T = \text{Spec } R$ with morphisms $U \to X$ and $T \to Y$ such that*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & Y \end{array}$$

commutes, there is at most one morphism $T \to X$ making the following diagram commute

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ T & \longrightarrow & Y. \end{array}$$