ALGEBRAIC GEOMETRY: MONDAY, APRIL 10

1. Properties and definitions

Definition 1.1. The **dimension** of a scheme is the dimension as a topological space. If $X \subset Y$ is an irreducible closed subset, the **codimension** of X in Y is the maximum length of a chain

$$X = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = Y$$

of irreducible closed subsets of Y.

To conclude this section, we'll define the fiber product of schemes. The fiber product of two S-schemes X and Y will be an S-scheme scheme $X \times_S Y$ with two morphisms of S-schemes, called projections, $X \times_S Y \to X$ and $X \times_S Y \to Y$, satisfying the following universal property: if Z is a scheme with maps $Z \to X$ and $Z \to Y$ that agree when composed with the maps $X \to S$ and $Y \to S$, then there is a unique morphism $Z \to X \times_S Y$ factoring $Z \to X$ and $Z \to Y$.

If X and Y are arbitrary schemes (not over a base scheme S), we take $X \times Y$ to mean $X \times_{\mathrm{Spec}\, \mathbb{Z}} Y$.

Proposition 1.2. *The fiber product exists and is unique up to unique isomorphism.*

Proof. We prove this only in the affine case. To prove the general case, one must do this construction locally and show that the resulting fiber products are compatible/glue together.

Suppose $S = \mathrm{Spec}\, R$, $X = \mathrm{Spec}\, A$, and $Y = \mathrm{Spec}\, B$, where A, B are R-algebras (meaning we have a map $R \to A$ and $R \to B$). Then, we claim that $\mathrm{Spec}\,(A \otimes_R B)$ has the required universal property.

First, an essential exercise:

Exercise 1.3. If Z is any scheme and A any ring, then $Hom_{\mathcal{S}ch}(Z, \mathrm{Spec}\, A)$ is in bijection with $Hom_{\mathcal{R}ing}(A, \mathcal{O}_Z(Z))$.

So, to give a map $Z \to \mathrm{Spec}\,(A \otimes_R B)$, it suffices to give a map $A \otimes_R B \to \mathcal{O}_Z(Z)$.

Suppose $Z \to X$ and $Z \to Y$ are morphisms of S-schemes (so they agree with the map to S). By the essential exercise, these are equivalent to morphisms $A \to \mathcal{O}_Z(Z)$ and $B \to \mathcal{O}_Z(Z)$, and the fact that they are morphisms of S-schemes says the map $R \to \mathcal{O}_Z(Z)$ factors through either of these maps. But, morphisms $A \to \mathcal{O}_Z(Z)$ and $B \to \mathcal{O}_Z(Z)$ that induce the same map on R are exactly morphisms $A \otimes_R B \to \mathcal{O}_Z(Z)$. Therefore, we get a unique map $Z \to \mathrm{Spec}\,(A \otimes_R B)$, as desired. □

We use fiber products to define *scheme-theoretic fibers* of morphisms of schemes.

Definition 1.4. Let $f : X \to Y$ be a morphism and $y \in Y$ be a point. The **residue field** of y is $k(y) = \mathcal{O}_{Y,y}/m_y$ where m_y is the unique maximal ideal in the local ring $\mathcal{O}_{Y,y}$. There is a natural morphism $\mathrm{Spec}\, k(y) \to Y$.

The **scheme-theoretic fiber** of f over y is $X_y := X \times_Y \mathrm{Spec}\, k(y)$.

Exercise 1.5. Show that the topological space of the scheme X_y is just $f^{-1}(y)$.

This definition allows us to consider fibers of morphisms as schemes. One important example of this occurs when we want to study families of schemes and how they deform.

Example 1.6. Let $X = \text{Spec } k[x, y, t]/(xy - t)$ and $Y = \text{Spec } k[t]$. There is a morphism $f : X \to Y$ from the ring map $k[t] \to k[x, y, t]/(xy - t)$.

If $a \in k$, $a \neq 0$, the fiber over the point $t = a$ is the hyperbola $X_a = \text{Spec } k[x, y]/(xy - a)$. If $a = 0$, we get two lines $X_0 = \text{Spec } k[x, y]/(xy)$. Geometrically, this is a family of smooth hyperbolas degenerating to the union of two lines.

How do we know this? If $t = a$, then the structure sheaf of Y at the point is $k[t]_{(t-a)}$, and the maximal ideal is the image of $(t - a)$ in the localization. So, $k(a) = k[t]_{(t-a)}/(t - a) \cong k$. Therefore,

$$X_a = X \times_Y k = \text{Spec } (k[x, y, t]/(xy - t) \otimes_{k[t]} k),$$

where k is a $k[t]$-module via the map $t \mapsto a$.

Using that $R[x_1, \dots, x_n]/I \otimes_R S \cong S[x_1, \dots, x_n]/IS[x_1, \dots, x_n]$ for any commutative ring R and R-algebra S, this tensor product ring is $k[t][x, y]/(xy-t) \otimes_{k[t]} k \cong k[x, y]/(xy-a)$. Therefore, the scheme-theoretic fiber over any point is $\text{Spec } k[x, y]/(xy - a)$.

2. Separated and Proper Morphisms

By now, we know that the Zariski topology is not Hausdorff, but we will introduce a condition called *separatedness* capturing a similar idea. We also may want compactness properties of our schemes, beyond just compactness of the topological space, to which we will introduce *properness.*

Definition 2.1. Let $f : X \to Y$ be a morphism of schemes. The **diagonal** is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with each projection is the identity.

Example 2.2. Suppose $X = \text{Spec } A$ and $Y = \text{Spec } B$ are affine schemes. Then, the diagonal is the map $\text{Spec } A \to \text{Spec } A \otimes_B A$ that gives the identity on the projections, and we know exactly what this map is. In terms of rings, the projections are $\text{Spec } A \otimes_B A \to \text{Spec } A$ given by $A \to A \otimes_B A$ where $a \mapsto a \otimes 1$ or $1 \otimes a$. Therefore, the diagonal is the map $A \otimes_B A \to A$ that induces the identity when composed with the projection, so it is given by $a \otimes a' = aa'$.

This morphism is called the diagonal because it gives the usual diagonal map on topological spaces.

Example 2.3. Let $X = \text{Spec } k[x] = \mathbb{A}^1_k \to Y = \text{Spec } k$. The diagonal is the map

$$X = \text{Spec } k[x] \to X \times_k X = \text{Spec } k[x] \otimes_k k[x] \cong \text{Spec } k[x_1, x_2] = \mathbb{A}^2_k$$

coming from the ring map

$$\begin{aligned} \phi : k[x_1, x_2] &\to k[x] \\ x_1 &\mapsto x \\ x_2 &\mapsto x. \end{aligned}$$

What is the image of any point in X? If $a \in \mathbb{A}^1$ is a closed point, then a corresponds to a maximal ideal $(x - a)$, and the image of this ideal is $\phi^{-1}(x - a) = (x_1 - a, x_2 - a)$, which is the closed point $(a, a) \in \mathbb{A}^2$. So, the image of the diagonal is the usual diagonal $x_1 = x_2$ in $\mathbb{A}^2$.