ALGEBRAIC GEOMETRY: FRIDAY, APRIL 7

1. Properties and definitions

Definition 1.1. A morphism of schemes $f : X \to Y$ is **locally of finite type** if there exists an open covering of Y by affine open subsets Spec B_i such that, for each i, f^{-1}Spec B_i can be covered by affine open subsets Spec A_{ij} where each A_{ij} is a finitely generated B_i algebra. The morphism is of **finite type** if each f^{-1}Spec B_i can be covered by finitely many affine open sets. In this case, we often say X is of finite type over Y. If $Y =$ Spec A is affine, we say X is of finite type of A.

Definition 1.2. A morphism of schemes $f : X \to Y$ is **finite** if there exists an open covering of Y by affine schemes Spec B_i such that, for each i, f^{-1}Spec $B_i =$ Spec A_i where A_i is a B_i-algebra that is a finitely generated B_i-module.

Recall that finite generation as a *module* is much stronger than finite generation as an *algebra.* Finite morphisms are so named because of the following property:

Exercise 1.3. Show that, if $f : X \to Y$ is finite, the preimage of any point $y \in Y$ is a finite set. (If the preimage of any point $y \in Y$ is finite, we say the map $f : X \to Y$ is quasi-finite.)

Example 1.4. Consider the map $k[t] \to k[x, y]/(y - x^2)$ given by $t \mapsto x$. This corresponds to the map of schemes Spec $k[x, y]/(y - x^2) \to$ Spec $k[t]$ mapping a closed point $(x - a, y - a^2)$ to $(t - a)$, i.e. $(a, a^2) \in k^2 \mapsto a \in k$. This is projecting the conic $y = x^2$ to its x-coordinate! Because $k[x, y]/(y - x^2) \cong k[x]$ is finitely generated as a module over $k[t]$ (via the map $t \mapsto x$), this is a finite morphism.

Similarly, we could project to the other axis. Consider the map $k[t] \to k[x, y]/(y - x^2)$ given by $t \mapsto y$. This maps the closed point $(x - a, y - a^2)$ to $(t - a^2)$, or $(a, a^2) \mapsto a^2$, projcting on the y-axis. Via the identification $k[x, y]/(y - x^2) \cong k[x]$, we have $t \mapsto x^2$, so $k[x]$ is a finitely generated $k[t]$-module, generated by 1 and x. So again, this is a finite morphism.

A word of caution, however: quasi-finite does not imply finite!

Example 1.5. Consider the map $k[t] \to k[x, y]/(xy - 1)$ given by $t \mapsto x$. This corresponds to the map of schemes Spec $k[x, y]/(xy - 1) \to$ Spec $k[t]$ mapping a closed point $(x - a, y - 1/a)$ to $(t - a)$, i.e. $(a, 1/a) \in k^2 \mapsto a \in k$. This is projecting the hyperbola to the x-axis. However, $k[x, y]/(xy - 1) \cong k[x, \frac{1}{x}]$ is *not* finitely generated as a module over $k[t]$. So, this is *not* a finite morphism. It is finitely generated as an algebra, so it is of finite type.

However, note that this map of schemes is not surjective; its image is

$$\mathbb{A}^1 - \{0\} = \text{Spec } k[t] - \{(t)\} = D(t) = \text{Spec } k[t]_t.$$

Because $k[t]_t \cong k[t, \frac{1}{t}]$, if we the ring map $k[t, \frac{1}{t}] \to k[x, y]/(xy - 1)$ where $t \mapsto x$ and $1/t \mapsto y$, this is an isomorphism of rings, hence the second is a finitely generated module over the first. Therefore, the map of schemes Spec $k[x, y]/(xy - 1) \to$ Spec $k[t]_t$ (projecting the hyperbola to its image in the x-axis) is finite.

Exercise 1.6. For each of the previous definitions, prove that it is equivalent to require the given property for *any* affine open subset of Y.

Definition 1.7. An **open subscheme** of a scheme X is a scheme U such that the topological space is isomorphic to an open subset of X and the structure sheaf $\mathcal{O}_U \cong \mathcal{O}_X|_U$.

An **open immersion** is a morphism $f : X \to Y$ that induces an isomorphism of X with an open subscheme of Y.

Definition 1.8. A **closed immersion** is a morphism $f : X \to Y$ that induces an isomorphism of the topological space of X with a closed subscheme of Y and such that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective. A **closed subscheme** is an equivalence class of closed immersions, where $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are equivalent if there is an isomorphism $g : X_1 \to X_2$ such that $f_1 = f_2 \circ g$.

A closed set (topologically) of Y may have several different scheme structures on it. Suppose $Y = \text{Spec } A$ and $X = \text{Spec } A/I$ for any ideal $I \subset A$. Then, X is a closed subscheme of Y (the proof will be outlined in the example below), which says for any closed subset $X = V(I) \subset Y$, X is a closed subscheme. In particular, any $X = V(I)$ is a closed subscheme but there is a *different* scheme structure on the topological space of X for each different I satisfying $X = V(I)$.

Example 1.9. For instance, let's say we wanted to say the topological space $\mathbb{A}^1$ was a closed subscheme of $\mathbb{A}^2$ (as one of the axes). To be specific, let's consider the map $f : \text{Spec } k[x] \to \text{Spec } k[x, y]$ associated to the ring map $\phi : k[x, y] \to k[x]$ where x maps to x and y maps to 0. This includes Spec $k[x] = \mathbb{A}^1$ as the x-axis: given any point a k (which corresponds to the maximal ideal $(x - a)$ in Spec $k[x]$), the image is $f(x - a) = \phi^{-1}(x - a) = (x - a, y)$, so corresponds to the point $(a, 0) \in k^2$.

What is the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$? Consider first the map on global sections, which is just $\phi : k[x, y] \to k[x]$. This is surjective by definition. To be a surjective morphism of sheaves, we need that $\mathcal{O}_{Y,f(p)} \to f_*\mathcal{O}_{X,p}$ is surjective for all $p \in X$. But, if p is a prime ideal in $k[x]$ and $q = f(p)$ is a prime in $k[x, y]$, these stalks are just the localizations $\phi : k[x, y]_q \to k[x]_p$, which is surjective because ϕ is. Therefore, in this way, $\mathbb{A}^1$ is a closed subscheme of $\mathbb{A}^2$.

However, the x-axis may be a closed subscheme in many other ways. For example, consider the map of schemes $f : \text{Spec } k[x, y]/(y^2) \to \text{Spec } k[x, y]$ given by the ring map $\phi : k[x, y] \to k[x, y]/(y^2)$. The topological space of Spec $k[x, y]/(y^2)$ is still the x-axis, and the map of sheaves is still surjective (because ϕ is), so this shows that the x-axis is also a closed subscheme of $\mathbb{A}^2$ with a different scheme structure.

Definition 1.10. If $X = V(p)$ is the vanishing of a prime ideal in $Y = \text{Spec } A$, then we say $X_n = V(p^n)$ is the ***n*th infinitesimal neighborhood of** X. (This has topologically the same structure as X, but a different sheaf of regular functions.)

Definition 1.11. If $X = V(I)$ is a closed subset of $Y = \text{Spec } A$, we often choose the 'simplest' scheme structure on X called the **reduced induced structure**, defined by $X = \text{Spec } A/J$ where $J = \cap p \text{prime}, p \subset X p$. For a non-affine scheme, we cover Y by open affine subsets and define this structure locally on X.

Example 1.12. In the previous example, where $X = V((y)) \subset \text{Spec } k[x, y]$, the primes contained in X are all primes containing y, and their intersection is just (y). So, the reduced induced structure on X is the simplest one $X = \text{Spec } k[x]$.

Definition 1.13. The **dimension** of a scheme is the dimension as a topological space. If $X \subset Y$ is an irreducible closed subset, the **codimension** of X in Y is the maximum length of a chain

$$X = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n = Y$$

of irreducible closed subsets of Y.