ALGEBRAIC GEOMETRY: WEDNESDAY, APRIL 5

1. Properties and definitions

We will explore many new definitions for schemes. The first few are just reminders from last time.

Definition 1.1. A scheme is **connected** if its topological space is connected. Otherwise, it is said to be **disconnected**.

Definition 1.2. A scheme is **irreducible** if its topological space is irreducible. Otherwise, it is said to be **reducible**.

Definition 1.3. A scheme X is **reduced** if, for every open subset U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Otherwise, it is said to be **non-reduced**.

Definition 1.4. A scheme X is **integral** if, for every open subset U, the ring $\mathcal{O}_X(U)$ is an integral domain.

Definition 1.5. A scheme X is **locally noetherian** if it can be covered by affine open sets Spec A_i where each A_i is a noetherian ring. A scheme X is **noetherian** if it is locally noetherian and quasi-compact, or equivalently X has a finite cover by Spec A_i where each A_i is noetherian.

Exercise 1.6. Show that, if a scheme X is noetherian, then the topological space is noetherian, but find an example to show the converse is false.

Exercise 1.7. If $X = \text{Spec } A$, show that X is quasi-compact.

Remark 1.8. In the definition of (locally) noetherian, we only require a property on *some* open cover. Many other properties will be defined similarly, but it is often convenient to know that the property exists for every open set (or every open cover). This is often true, and we will illustrate it in this example. In the context of the previous exercise, this will say that $X = \text{Spec } A$ is noetherian if and only if A is noetherian.

Proposition 1.9. *A scheme X is locally noetherian if and only if for every affine open set $U \subset X$, $U = \text{Spec } A$, the ring A is noetherian.*

Proof. The only if direction follows from the definition.

So, assume X is locally noetherian and let $U = \text{Spec } A$ be an affine open subset. Let $X = \cup \text{Spec } B_i$ be an open cover such that each B_i is noetherian. Then, $U = \text{Spec } A \cap (\cup \text{Spec } B_i)$. Because each Spec $A \cap$ Spec B_i is an open subset of Spec B_i and the open sets $D(f) = \text{Spec } B_{i,f}$ form a base for the topology, we may assume Spec A is a union of Spec $B_{i,f}$. Furthermore, each $B_{i,f}$ is noetherian because B_i was noetherian.

So, we have shown that Spec A has an open cover by locally noetherian affine subsets. Let $U = \text{Spec } B$ be one of these open subsets of X, and let $f \in A$ be an element such that $D(f) \subset U = \text{Spec } B$. Because Spec $B \subset$ Spec A, there is a map $\phi : A \to B$, and let g be the image of f in B. Then, $D(f) = D(g)$ so $A_f \cong B_g$, which says A_f is noetherian. Because Spec A is quasi-compact, we can write Spec $A = \cup_{i=1}^n \text{Spec } A_{f_i}$ for some collection of $f_i \in A$ with each A_{f_i} noetherian.

Because Spec $A = \cup_{i=1}^n \text{Spec } A_{f_i} = \cup_{i=1}^n D(f_i)$, we must have $A = (f_1, \ldots, f_n)$ (otherwise, $(f_1, \ldots, f_n)$ would be contained in some maximal ideal m, and by definition, $m \in \text{Spec } A$, but

$m \notin D(f_i)$ for any i). Therefore, we have shown that $A = (f_1, \dots, f_n)$ and each A_{f_i} is noetherian. We wish to conclude that A is noetherian. This has a purely algebraic proof, so you may use it as a fact from algebra. If you want to see the proof, it's included below.

Let $I \subset A$ be any ideal, and let $\phi_i : A \to A_{f_i}$ be the localization map. We first prove that $I = \cap_{i=1}^n \phi_i^{-1}(\phi_i(I) \cdot A_{f_i})$. The inclusion $\subset$ is clear, so assume $b \in A$ is an element of the left side. Write $\phi_i(b) = a_i/f_i^r$, where $a_i \in I$. (We may assume that r is constant for all i by increasing if necessary.) This means $b = a_i/f_i^r \in A_{f_i}$, or $f_i^{m_i}(f_i^r b - a_i) = 0 \in A$ (and by increasing m_i, we may assume all $m_i = m$). Therefore, $f_i^{r+m} b = f_i^m a \in I$ for each i. Because $A = (f_1, \dots, f_n)$, it is also true that $A = (f_1^N, \dots, f_n^N)$ for any N, so for $N = r + m$, it means there is a linear combination $b = \sum_{i=1}^n c_i f_i^N b$, which implies $b \in I$.

Now, suppose $I_1 \subset I_2 \subset I_3 \subset \dots$ is an ascending chain of ideals. Then, $\phi_i(I_1)\cdot A_{f_i} \subset \phi_i(I_2)\cdot A_{f_i} \subset \dots$ is an ascending chain in A_{f_i}, which is noetherian, so eventually stabilizes. This is true for each i, so the chain $\phi_i^{-1}(\phi_i(I_1) \cdot A_{f_i}) \subset \phi_i^{-1}(\phi_i(I_2) \cdot A_{f_i}) \subset \dots$ eventually stabilizes. By the previous paragraph, $I_1 \subset I_2 \subset \dots$ eventually stabilizes and thus A is noetherian. □

Corollary 1.10. An affine scheme $X = \text{Spec } A$ is noetherian if and only if A is noetherian.

Definition 1.11. A morphism of schemes $f : X \to Y$ is **locally of finite type** if there exists an open covering of Y by affine open subsets Spec B_i such that, for each i, $f^{-1}\text{Spec } B_i$ can be covered by affine open subsets Spec A_{ij} where each A_{ij} is a finitely generated B_i algebra. The morphism is of **finite type** if each $f^{-1}\text{Spec } B_i$ can be covered by finitely many affine open sets. In this case, we often say X is of finite type over Y. If $Y = \text{Spec } A$ is affine, we say X is of finite type of A.

Example 1.12. If $X = \text{Spec } k[x_1, \dots, x_n]/I$ is the scheme corresponding to an affine variety $V(I)$, then X has a map to $Y = \text{Spec } k$ from the inclusion of rings $k \to k[x_1, \dots, x_n]/I$. Because the preimage of $Y = \text{Spec } k$ is $X = \text{Spec } k[x_1, \dots, x_n]/I$ and $k[x_1, \dots, x_n]/I$ is a finitely generated k-algebra, X is of finite type over k. In general, the scheme corresponding to any variety over k is of finite type over k.

Example 1.13. Let $x \in X$ be a point of an affine variety with sheaf $\mathcal{O}_{X,x}$. Then, Spec $\mathcal{O}_{X,x}$ is an integral noetherian scheme (because $\mathcal{O}_{X,x}$ is a domain and a noetherian ring) but it is not in general of finite type over k. For example, if $X = \mathbb{A}^1$ and $x = 0$, then $\mathcal{O}_{X,x} = k[x]_{(x)}$. This means we invert everything outside of the ideal x, and there are infinitely many of these reciprocals $1/f(x)$ that are linearly independent. (For example, $1/(x - a)$ is in this ring for any $a \in k$, $a \neq 0$, and these are all independent elements.) So, $k[x]_{(x)}$ is not finite type over k.

Definition 1.14. A morphism of schemes $f : X \to Y$ is **finite** if there exists an open covering of Y by affine schemes Spec B_i such that, for each i, $f^{-1}\text{Spec } B_i = \text{Spec } A_i$ where A_i is a B_i-algebra that is a finitely generated B_i-module.

Recall that finite generation as a *module* is much stronger than finite generation as an *algebra.*

Example 1.15. Suppose $X = \text{Spec } k[x]$ and $Y = \text{Spec } k$. This is not finite: $k[x] = k \oplus kx \oplus kx^2 \oplus \dots$ is not finitely generated as a module over k.

Exercise 1.16. For each of the previous definitions, prove that it is equivalent to require the given property for *any* affine open subset of Y.

Finite morphisms are so named because of the following property:

Exercise 1.17. Show that, if $f : X \to Y$ is finite, the preimage of any point $y \in Y$ is a finite set. (If the preimage of any point $y \in Y$ is finite, we say the map $f : X \to Y$ is quasi-finite.)