ALGEBRAIC GEOMETRY: MONDAY, APRIL 3

1. Properties and definitions

We will explore many new definitions for schemes.

Definition 1.1. A scheme is **connected** if its topological space is connected. Otherwise, it is said to be **disconnected**.

Definition 1.2. A scheme is **irreducible** if its topological space is irreducible. Otherwise, it is said to be **reducible**.

Example 1.3. Let $X = \text{Spec } k[x,y]/(xy)$. This is reducible: $X = V((x)) \cup V((y))$. (Why? $X = \{p \in \text{Spec } k[x,y]/(xy)\}$, which means X is the set of prime ideals in $k[x,y]$ containing (xy), a prime p contains xy if and only if p contains x or y, i.e. $p \in V((x))$ or $p \in V((y))$.)

Geometrically, what is going on? The prime ideals of Spec $k[x,y]$ are (0), the maximal ideals $\{(x-a, y-b) \mid a, b \in k\}$, or (f) where f is an irreducible polynomial. Because the prime ideals of Spec $k[x,y]/(xy)$ are the ideals of Spec $k[x,y]$ that contain xy, we can say

$$\text{Spec } k[x,y]/(xy) = \{\{(x, y-b) \mid b \in k\}, \{(x-a, y) \mid a \in k\}, (x), (y)\}.$$

Note that $V((x)) = \{\{(x, y-b) \mid b \in k\}, (x)\}$ and $V((y)) = \{\{(x-a, y) \mid a \in k\}, (y)\}$, so we see exactly that

$$\text{Spec } k[x,y]/(xy) = V((x)) \cup V((y)).$$

Graphing, the maximal ideals of $V((x)) = \{\{(x, y-b) \mid b \in k\}, (x)\}$ correspond to the points in $\mathbb{A}^2$ with $x = 0$ and $y = b$, and then (x) corresponds to the generic point (which lives 'everywhere' on $V((x))$), so the geometric object is the line $(x = 0)$ in $\mathbb{A}^2$. Similarly, $V((y))$ is the geometric object $(y = 0)$ in $\mathbb{A}^2$, so the picture consists of two lines, each closed in the Zariski topology, which tells us the space is not irreducible.

Definition 1.4. A scheme X is **reduced** if, for every open subset U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements. Otherwise, it is said to be **non-reduced**.

Definition 1.5. A scheme X is **integral** if, for every open subset U, the ring $\mathcal{O}_X(U)$ is an integral domain.

Exercise 1.6. If $X = \text{Spec } A$ is an affine scheme, then X is reduced if and only if A has no nilpotents, and integral if and only if A is integral. (Hint: write open sets U as unions of localizations of A, and relate nilpotents/zero divisors in localizations to those in A.)

Example 1.7. Going back to the previous example $X = \text{Spec } k[x,y]/(xy)$, the ring $k[x,y]/(xy)$ is not a domain ($x, y \neq 0$ but $xy = 0$), so X is not integral, but it has no nilpotents, so X is reduced.

Example 1.8. Let $X = \text{Spec } k[x]/(x^2)$. The ring $k[x]/(x^2)$ has a nilpotent element x, so X is not reduced.

Proposition 1.9. *X is integral if and only if it is reduced and irreducible.*

Proof. If X is integral, then $\mathcal{O}_X(U)$ is an integral domain, which means $\mathcal{O}_X(U)$ has no nilpotents, so X is reduced. If X is reducible, then there exist disjoint open subsets $U_1, U_2 \subset X$, but because they are disjoint, $\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \times \mathcal{O}(U_1)$, so $\mathcal{O}(U_1 \cup U_2)$ is not an integral domain! (For example, $(1,0) \cdot (0,1) = (0,0)$.) Therefore, X integral also implies X irreducible.

For the converse, we prove it only in the affine case. (Exercise: prove it in the general case!) Suppose that $X = \operatorname{Spec} A$ is reduced and irreducible, and consider $f, g \in A$ with $fg = 0$. We must show that A is integral, meaning either f or g is 0. Let $Y = V((f))$ and $Z = V((g))$. Because $fg = 0$, $V((f)) \cup V((g)) = V((fg)) = V((0)) = X$. Because X was irreducible, we must have either $V((f)) = X$ or $V((g)) = X$. Without loss of generality, suppose $X = V((f))$. Therefore, $V((f)) = \operatorname{Spec} A = V((0))$, so $f \in \sqrt{(0)}$. This means f is nilpotent, but X was reduced, so this implies $f = 0$. Therefore, X is integral. $\square$