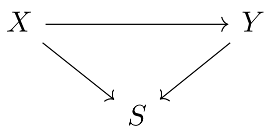
ALGEBRAIC GEOMETRY: FRIDAY, MARCH 31

1. Introduction to Schemes, Conclusion

Another categorical correspondence we have mentioned (but not yet proven) is the following (which will make use of the following definition):

Definition 1.1. Let S be a scheme. The category of schemes over S is the category whose objects are schemes X with a morphism $X \to S$ and morphisms $X \to Y$ are morphisms compatible with the given maps to S, i.e. a morphism $X \to Y$ is a map making the following diagram commute:



If $S = \text{Spec } A$, we say that a scheme over S is an A-scheme or scheme over A.

Theorem 1.2. *Suppose k is an algebraically closed field. Let $\mathcal{V}ar_k$ be the category of varieties over k and $\mathcal{S}ch_k$ be the category of schemes over k. There is a fully faithful functor $t : \mathcal{V}ar_k \to \mathcal{S}ch_k$. Furthermore, if $V \mapsto t(V)$, then V is homeomorphic to the closed points of the topological space of the scheme $t(V)$, and the sheaf of regular functions on V is the restriction of the structure sheaf of $t(V)$ under this homeomorphism.*

In other words, this theorem says we can think of varieties as a subcategory of the category of schemes via the functor t.

Proof. Define the functor t as follows: on objects, if X is a topological space, then $t(X)$ is the set of non-empty irreducible closed subsets of X. Observations: if $Y \subset X$, then $t(Y) \subset t(X)$; $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$; $t(\cap Y_i) = \cap t(Y_i)$; which together imply that there is a topology on $t(X)$ by letting the closed sets be $t(Y)$, where Y is a closed subset of X.

For morphisms, if $f : X_1 \to X_2$ is continuous, let $t(f) : t(X_1) \to t(X_2)$ send an irreducible closed subset to the closure of its image in X_2. This shows that t is a functor on topological spaces.

Also, note that there is a continuous map $i : X \to t(X)$ given by $p \mapsto \overline{\{p\}}$. To define the structure sheaf, we let $\mathcal{O}_{t(X)} = i_* \mathcal{O}_X$.

Now, to prove the equivalence, let k be an algebraically closed field and V a variety over k with sheaf of regular functions $\mathcal{O}_V$. We will show that $(t(V), i_*\mathcal{O}_V)$ is a scheme. Since every variety is covered by affine varieties, it is sufficient to prove this for V an affine variety with affine coordinate ring A. In this case, we will show that $(t(V), i_*\mathcal{O}_V)$ is just $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

Note that $(V, \mathcal{O}_V)$ is a locally ringed space, and we can define a morphism of locally ringed spaces $(f, f^\#) : (V, \mathcal{O}_V) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ as follows: if $v \in V$, let $f(v) = m_v = \{a \in A \mid a(v) = 0\}$. Because the points of V are in one-to-one correspondence with the maximal ideals of A (= closed points of Spec A), this is a bijection from V to the closed points of Spec A, and it is a homeomorphism onto its image. To define $f^\#$, given any $s \in \mathcal{O}_{\text{Spec } A}(U)$, this gives a regular function $s_V : f^{-1}(U) \to k$ by, for any $v \in f^{-1}(U)$, $s_V(v) = \overline{s(f(v))}$, where $f(v) = m_v$, $s(m_v) \in \mathcal{O}_{\text{Spec } A, m_v} \cong A_{m_v}$, and $\overline{s(f(v))}$ is the image in the residue field $A_{m_v}/m_v \cong k$. In fact, this gives an isomorphism $\mathcal{O}_{\text{Spec } A}(U) \cong \mathcal{O}_V(f^{-1}(U))$, as any regular function $s_V \in \mathcal{O}_V(f^{-1}(U))$ is $s : f^{-1}(U) \to k$ such that, at any point v, this is locally a quotient a/b of elements in A with nonzero denominator at v, meaning (near v) we can interpret $s = a/b$ as a map to $\cup_{p\in U} A_p$, which is a regular function on $\mathcal{O}_{\text{Spec } A}(U)$.

Lastly, recall that the irreducible closed subsets of V are exactly the prime ideals of A, which are the points of $t(V)$ and Spec A, so (using the definition of the topology on Spec A and $t(V)$), $(\text{Spec } A, \mathcal{O}_{\text{Spec } A}) \cong (t(V), i_*\mathcal{O}_V)$. Furthermore, this is a scheme over k because there is a ring homomorphism $k \to i_*\mathcal{O}_V(t(V)) = \mathcal{O}_V(V)$ given by $\lambda \in k$ goes to the constant function, which gives a map $t(V) \to \text{Spec } k$.

Now, to conclude the proof, we leave it as an exercise to verify that, if V and W are varieties over k, $\text{Hom}_{\mathcal{V}ar_k}(V, W) \to \text{Hom}_{\mathcal{S}ch_k}(t(V), t(W))$ is bijective, which is what is meant by 't is fully faithful.' (Possible idea: reduce to the affine case, and show both of these are in bijection with Hom between two rings?) □

Definition 1.3. With the previous theorem, we define $\mathbb{A}^n_k$ as a scheme to be $\mathbb{A}^n_k = \text{Spec } k[x_1, \ldots, x_n]$.

2. Projective Schemes

Analagous to our definition of projective variety, we can define a projective scheme. Let $S = \oplus_{d\geq 0} S_d$ be a graded ring. Let $S_+ = \oplus_{d>0} S_d$. This is often called the *irrelevant ideal.*

Definition 2.1. Proj S is the set of all homogeneous prime ideals in S that do not contain S_+. The topology on Proj S is obtained by taking the closed subsets to be $V(I) = \{p \in \text{Proj } S \mid I \subset p\}$ where I is a homogeneous ideal of S.

Exercise 2.2. Check that this is a topology on Proj S.

Definition 2.3. For $p \in \text{Proj } S$, let $S_{(p)}$ denote the elements of degree 0 in the localization $T^{-1}S$, where T is the multiplicatively closed subset of homogeneous elements in S not in p. The structure sheaf of Proj S is defined as, for any $U \subset \text{Proj } S$, $\mathcal{O}_{\text{Proj } S}(U)$ is the set of functions $s : U \to \cup S_{(p)}$ such that, for each $p \in U$, there is a neighborhood V of p and $s|_V = a/b$ for homogeneous $a, b \in S$, $\deg a = \deg b$, and $b \notin q$ for all $q \in V$.

The objects and their structure sheaf behave analogously to what we saw for projective varieties. We have the following proposition whose proof is similar to that of varieties.

Proposition 2.4. *Let S be a graded ring.*

(1) For any $p \in \text{Proj } S$, $\mathcal{O}_p \cong S_{(p)}$.

(2) For any homogeneous $f \in S_+$, let $D_+(f) = \{p \in \text{Proj } S \mid f \notin p\}$. This is open in Proj S *and* Proj S *has an open cover by sets of this form. Furthermore, $D_+(f) \cong \text{Spec } S_{(f)}$ (as schemes).*

Corollary 2.5. Proj S is a scheme.

Proof. Part (1) of the proposition tells us Proj S is a locally ringed space, and part (2) tells us it is locally isomorphic to an affine scheme. □

Definition 2.6. We define $\mathbb{P}^n_k = \text{Proj } k[x_0, \ldots, x_n]$ to be projective n-space over k. (Note: we could generalize–we say projective n-space over any ring A is Proj $A[x_0, \ldots, x_n]$.)

Note: the previous proposition recovers what we already saw for affine varieties. In particular: Because $\mathbb{P}^n = \text{Proj } S$ where $S = k[x_0, \ldots, x_n]$, and if $f = x_0$, then this says $D_+(x_0) \subset \mathbb{P}^n$ is isomorphic to the affine scheme Spec $S_{(x_0)}$, but $S_{(x_0)} = k[x_1/x_0, \ldots, x_n/x_0]$, so this is just $\mathbb{A}^n$ (in exactly the same way as $D(x_0)$ was $\mathbb{A}^n$ for varieties).