ALGEBRAIC GEOMETRY: WEDNESDAY, MARCH 29

1. Introduction to Schemes, Part IV

Definition 1.1. A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for each $p \in X$, $\mathcal{O}_{X,p}$ is a local ring.

A **morphism** of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y such that, for all $p \in X$, the induced map $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism.

Definition 1.2. An **isomorphism** of locally ringed spaces is a morphism with two sides inverse, or equivalently a pair $(f, f^\#)$ where f is a homeomorphism of topological spaces and $f^\#$ an isomorphism of sheaves.

Definition 1.3. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $(\text{Spec } A, \mathcal{O})$ for some ring A. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that, for every point $p \in X$, there is a neighborhood U of p such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

From the local definition of scheme, we can make schemes that we couldn't obtain as varieties.

Example 1.4. Suppose X_1, X_2 are schemes with U_1, U_2 are open subsets. If there is an isomorphism $\phi : (U_1, \mathcal{O}_{X_1}|_{U_1}) \to (U_2, \mathcal{O}_{X_2}|_{U_2})$ is an isomorphism of locally ringed spaces, we can *glue* X_1 and X_2 together along the U_i. Let $X = X_1 \cup X_2/\sim$, where $x_1 \sim x_2$ if $x_2 = \phi(x_1)$ with the quotient topology. Let $j_i : X_i \to X$, $i = 1, 2$ denote the inclusions. Let $\mathcal{O}_X$ be the sheaf defined on open sets by

$$\mathcal{O}_X(V) = \{(s_1, s_2) \mid s_i \in \mathcal{O}_{X_i}(j_i^{-1}(V)), \phi(s_1|_{j_1^{-1}(V)\cap U_1}) = s_2|_{j_2^{-1}(V)\cap U_2}$$

(so, the sections are sections of X_1 and X_2 that "agree" via ϕ on the overlap).

This is a scheme: every point has an open neighborhood that lies in X_1 or X_2, which were schemes, so every point has a neighborhood which is affine.

For a specific example, let $X_1 = X_2 = \mathbb{A}_k^1$ (where $\mathbb{A}_k^1 = \text{Spec } k[x]$). Let $U_1 = U_2 = \mathbb{A}_k^1 - p$, where p is the point corresponding to $(x) \in \text{Spec } k[x]$ ('the origin'), and let $\phi : U_1 \to U_2$ be the identity. We can glue, and then X becomes something that looks mostly like $\mathbb{A}_k^1$, but has two origins. This is not affine! We will see how to prove this later, but if you'd like, you can try to prove it with what we know now.

Proposition 1.5. *Let A and B be rings.*

(1) If $\phi : A \to B$ is a ring homomorphism, then ϕ induces a morphism

$$(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A}).$$

(2) If $(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is any morphism of locally ringed spaces, then it is induced by a ring homomorphism $\phi : A \to B$.

Example 1.6. Let $A = k[x, y]/(y - x^2)$ and let $B = k[t]$. Consider the map $\phi : A \to B$ given by $\phi(x) = t$ and $\phi(y) = t^2$.

What is the induced map $(\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$? Reminder: $\phi : A \to B$ induces the map $f : \text{Spec } B \to \text{Spec } A$ given by $f(p) = \phi^{-1}(p)$.

Note that $\text{Spec } B = \{(0), \{(t - a) \mid a \in k\}\}$ and $\text{Spec } A = \{(0), \{(x - a, y - a^2) \mid a \in k\}\}$ (by the correspondence of ideals in $k[x, y]$ containing $(y - x^2)$ and ideals in $k[x, y]/(y - x^2)$).

By definition of ϕ, $\phi^{-1}(0) = (0)$, so the generic point of Spec B maps to the generic point of Spec A.

For the closed points, by definition, $\phi^{-1}(t-a) = (x-a, y-a^2)$, so the map takes $(t-a)$ (thought of as the closed point $a \in k$) to $(x-a, y-a^2)$ (to the closed point (a, a^2)).

Now, for the sheaves/regular functions, we have a map $f^\# : \mathcal{O}_{\text{Spec } A} \to f_* \mathcal{O}_{\text{Spec } B}$ given by $s \mapsto \phi \circ s \circ f$. For example, suppose we are looking on the open set $U \subset \text{Spec } A$ where $U = D(x)$, the primes that don't contain x. This is every prime other that (x, y), i.e. it is the complement of the point $(0,0)$ in the graph of $y - x^2 = 0$. A regular function on this open set is, by a previous proposition, an element of A_x, so regular functions are allowed to have x in the denominator. For example, the function $s = (2x^2 - 3y^3 + 1)/x$ is an element of $\mathcal{O}_{\text{Spec } A}(D(x))$. On this open set, $f^\#$ is a map $\mathcal{O}_{\text{Spec } A}(D(x)) \to \mathcal{O}_{\text{Spec } B}(f^{-1}D(x))$. The preimage of $D(x) = \{p \in \text{Spec } A \mid x \notin p\}$ is the set

$$\{q \in \text{Spec } B \mid x \notin f(q)\} = \{q \in \text{Spec } B \mid x \notin \phi^{-1}q\} =$$
$$\{q \in \text{Spec } B \mid \phi(x) \notin q\} = \{q \in \text{Spec } B \mid t \notin q\} = D(t)$$

so $f^\#$ is a map $\mathcal{O}_{\text{Spec } A}(D(x)) \to \mathcal{O}_{\text{Spec } B}(D(t))$, which is just $A_x \to B_t$, given by $f^\#(s) = \phi(s)$, i.e. $f^\#((2x^2 - 3y^3 + 1)/x) = (2t^2 - 3t^6 + 1)/t$.

Another way to state the proposition: if X, Y are affine schemes, then $\text{Mor}(X, Y)$ (morphisms of locally rings spaces) is in bijection with $\text{Mor}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$ (morphisms of rings), and vice versa: if A, B are rings, $\text{Mor}(A, B)$ is in bijection with $\text{Mor}(\text{Spec } B, \text{Spec } A)$.

Theorem 1.7. *The category of affine schemes is equivalent to the opposite category of rings, with equivalence given by the functor $X \mapsto \mathcal{O}_X(X)$ (and inverse functor $A \mapsto \text{Spec } A$).*

Another categorical correspondence we have mentioned (but not yet proven) is the following (which will make use of the following definition):

Definition 1.8. Let S be a scheme. The category of schemes over S is the category whose objects are schemes X with a morphism $X \to S$ and morphisms $X \to Y$ are morphisms compatible with the given maps to S, i.e. a morphism $X \to Y$ is a map making the following diagram commute:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \qquad \swarrow & \\ & S & \end{array}$$

If $S = \text{Spec } A$, we say that a scheme over S is an A-scheme or scheme over A.

Theorem 1.9. *Suppose k is an algebraically closed field. Let $\mathcal{V}ar_k$ be the category of varieties over k and $\mathcal{S}ch_k$ be the category of schemes over k. There is a fully faithful functor $t : \mathcal{V}ar_k \to \mathcal{S}ch_k$. Furthermore, if $V \mapsto t(V)$, then V is homeomorphic to the closed points of the topological space of the scheme $t(V)$, and the sheaf of regular functions on V is the restriction of the structure sheaf of $t(V)$ under this homeomorphism.*

In other words, this theorem says we can think of varieties as a subcategory of the category of schemes via the functor t.

We will prove this next time, but first, what is the functor t going to be? If X is a topological space, then $t(X)$ will be the set of non-empty irreducible closed subsets of X.

Why are we using this functor? Imagine that X is an affine variety. Then, it comes with some homogeneous coordinate ring $A(X)$, and we know that the irreducible closed subsets of X correspond to the *prime ideals* of $A(X)$. In other words, $t(X)$ will be the set of prime ideals in $A(X)$, so $t(X) = \text{Spec } A(X)$. Using the compatibility of regular functions on a variety and regular functions on an affine scheme (both defined as quotients of elements of the ring A), this will show that $t(X)$ is an affine scheme as we want.