ALGEBRAIC GEOMETRY: MONDAY, MARCH 27

1. Introduction to Schemes, Part III

Last time, we proved the following:

Proposition 1.1. *Let $X = \text{Spec } A$ and $\mathcal{O}_X$ be the structure sheaf.*

(1) For any point $p \in \text{Spec } A$ (= prime ideal of A), the stalk of $\mathcal{O}_X$ at p is $\mathcal{O}_p \cong A_p$.

(2) If $U = D(f)$, $f \in A$, is an open set in the base for the topology, then $\mathcal{O}_X(U) \cong A_f$. In particular, $X = D(1)$, so $\mathcal{O}_X(X) \cong A$.

Now, we have taken any ring and associated to it (1) a topological space and (2) a sheaf of rings, constructed essentially from the data of the ring we started with. This pair of objects will ultimately be called a scheme, but we have a few more definitions before getting there.

Definition 1.2. A **ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A **morphism** of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y.

A scheme will be something slightly stronger. First, a reminder/definition: suppose A and B are local rings and $\phi : A \to B$ a homomorphism. This map is called a **local homomorphism** if $\phi^{-1} m_B = m_A$.

Definition 1.3. A **locally ringed space** is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X such that for each $p \in X$, $\mathcal{O}_{X,p}$ is a local ring.

A **morphism** of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on Y such that, for all $p \in X$, the induced map $f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism.

Note: the induced map $f_p^\#$ is defined by the composition

$$\mathcal{O}_{Y,f(p)} = \lim_{f(p)\in V} \mathcal{O}_Y(V) \xrightarrow{f^\#(V)} \lim_{f(p)\in V} \mathcal{O}_X(f^{-1}(V)) \longrightarrow \lim_{p\in U} \mathcal{O}_X(U) = \mathcal{O}_{X,p}.$$

Corollary 1.4 (of the first proposition)**.** For any ring A, $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a locally ringed space (we proved that $\mathcal{O}_{\text{Spec } A,p} \cong A_p$, so the stalks are local rings).

Definition 1.5. An **isomorphism** of locally ringed spaces is a morphism with two sides inverse, or equivalently a pair $(f, f^\#)$ where f is a homeomorphism of topological spaces and $f^\#$ an isomorphism of sheaves.

Definition 1.6. An **affine scheme** is a locally ringed space $(X, \mathcal{O}_X)$ that is isomorphic to $(\text{Spec } A, \mathcal{O})$ for some ring A. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ such that, for every point $p \in X$, there is a neighborhood U of p such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Proposition 1.7. *Let A and B be rings.*

(1) If $\phi : A \to B$ is a ring homomorphism, then ϕ induces a morphism

$$(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A}).$$

(2) If $(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is any morphism of locally ringed spaces, then it is induced by a ring homomorphism $\phi : A \to B$.

Proof. For (1), suppose $\phi : A \to B$. Define $f : \text{Spec } B \to \text{Spec } A$ by $f(p) = \phi^{-1}(p)$. By definition, $f^{-1}(V(I)) = V(\phi(I))$, so f is continuous. For any point $p \in \text{Spec } B$, we have a map of local rings $\phi_p : A_{\phi^{-1}(p)} \to B_p$ which gives a map $f^\# : \mathcal{O}_{\text{Spec } A}(V) \to \mathcal{O}_{\text{Spec } B}(f^{-1}(V))$ by definition of $\mathcal{O}$ (a section is a function $s : V \to \cup_{q \in V} A_q$, and which we can precompose with $f : f^{-1}(V) \to V$ and postcompose with ϕ_p to get a map $\phi \circ s \circ f : f^{-1}(V) \to \cup_{p \in f^{-1}(V)} B_p$, which is a section of $\mathcal{O}_{\text{Spec } B}(f^{-1}(V))$). So, we have a morphism of sheaves $f^\# : \mathcal{O}_{\text{Spec } A} \to f_* \mathcal{O}_{\text{Spec } B}$. The maps on stalks are just the maps ϕ_p, which are local homomorphisms by definition. So, ϕ induced a morphism of locally ringed spaces.

For (2), suppose we have $(f, f^\#) : (\text{Spec } B, \mathcal{O}_{\text{Spec } B}) \to (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$. The map $f^\#$ gives a map

$$\phi := f^\#(\text{Spec } A) : \mathcal{O}_{\text{Spec } A}(\text{Spec } A) \cong A \to \mathcal{O}_{\text{Spec } B}(\text{Spec } B) \cong B.$$

Now, we just need to show that the morphism induced by ϕ is exactly $(f, f^\#)$. For any $p \in \text{Spec } B$, there is an induced local homomorphism $f_p^\# : \mathcal{O}_{\text{Spec } A, f(p)} \to \mathcal{O}_{\text{Spec } B, \phi_p}$, which is just $f_p^\# : A_{f(p)} \to B_p$, and this commutes with ϕ, the global map induced by $f^\#$. In other words, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(p)} & \xrightarrow{f_p^\#} & B_p. \end{array}$$

Because $f_p^\#$ is a local homomorphism, we must have $\phi^{-1}(p) = f(p)$, so f coincides with the map $\text{Spec } B \to \text{Spec } A$ induced by ϕ, which also implies $f^\#$ is induced by ϕ. □