ALGEBRAIC GEOMETRY: FRIDAY, MARCH 24

1. Introduction to Schemes, Part II

Definition 1.1. Suppose $X = \text{Spec } A$. The **structure sheaf** of X is the sheaf $\mathcal{O}_X$ defined as follows. For $p \in \text{Spec } A$, let A_p be the localization of A at the prime p. For any open set U, let $\mathcal{O}_X(U)$ be the set of functions $s : U \to \cup_{p \in U} A_p$ such that in a neighborhood V of each $p \in U$, $s|_V = \frac{f}{g}$, where $f, g \in A$, $g \notin q$ for any $q \in V$.

In other words, we are saying that the 'regular functions' on Spec A are locally quotients f/g of elements of A, and 'the denominator being nonzero' at a point $p \in \text{Spec } A$ means that $g \notin p$ so that $f/g \in A_p$.

Now, some properties of $\mathcal{O}_X$. *Notation: if $f \in A$, let $D(f)$ be the complement of* $V((f))$. By definition:

$$V((f)) = \{p \in \text{Spec } A \mid f \in p\}$$

and

$$D(f) = \{p \in \text{Spec } A \mid f \notin p\}.$$

Last time we proved:

Lemma 1.2. *The open sets $D(f)$ form a base for the topology on* Spec A.

What does $\mathcal{O}_X$ actually look like? The following proposition will tell us.

Proposition 1.3. *Let $X = \text{Spec } A$ and $\mathcal{O}_X$ be the structure sheaf.*

(1) For any point $p \in \text{Spec } A$ (= prime ideal of A), the stalk of $\mathcal{O}_X$ at p is $\mathcal{O}_p \cong A_p$.
(2) If $U = D(f)$, $f \in A$, is an open set in the base for the topology, then $\mathcal{O}_X(U) \cong A_f$. In particular, $X = D(1)$, so $\mathcal{O}_X(X) \cong A$.

Proof. Some reminders before we begin the proof: if S is a multiplicatively closed subset of A, then

$$S^{-1}A = \{\frac{a}{s} \mid a \in A, s \in S\}/\sim$$

where two elements are equivalent if:

$$a/s \sim b/t \iff \text{there exists } r \in S \text{ such that } r(at - bs) = 0.$$

If p is a prime ideal in A, let $S_p = \{a \in A \mid a \notin p\}$ and define $A_p := S_p^{-1}A$ (we invert the elements of A outside of p). If f is an element of A, then let $S_f = \{1, f, f^2, \dots\}$ and define $A_f := S_f^{-1}A$.

First we prove (1).

Define a map $\phi : \mathcal{O}_p \to A_p$ by sending any section s to its value $s_p \in A_p$.

Explicitly: we are saying take a section $s \in \mathcal{O}_p$, which we know is an element of $\mathcal{O}_X U$ for some open set U containing p, and on that open set has the expression $s = f/g$ for $f, g \in A$, where $g \notin p$. The element f/g then is an element of A_p. So, this map is taking s to the (equivalence class of the) representative $f/g \in A_p$.

Almost by definition, this is surjective: given any $f/g \in A_p$, on the open set $U = D(g)$, this is a section of $\mathcal{O}_X(U)$, and the definition of the map says f/g has image f/g. It is also injective: suppose $s, t \in \mathcal{O}_p$ are two elements with the same value at p. Then, by choosing a sufficiently small neighborhood U of p, we can assume $s = f/g$ and $t = h/j$ for elements $f, g, h, j \in A$ with

$g, j \notin p$. Now, we want to prove that s and t are equal in the stalk $\mathcal{O}_p$, which means that there is some open set V containing p where $s|_V = t|_V$. On U, since $s = f/g$ and $t = h/j$ these have the same image in A_p, that means $f/g = h/j$ in A_p, so there exists some $a \in A$, $a \notin p$, with $f/g = ah/aj$, or $a(fj - gh) = 0 \in A$. This implies, for any q such that $a, g, j \notin q$, it is also true that $f/g = h/j \in A_q$. In other words, on the open set $V = D(a) \cap D(g) \cap D(j)$ (which is the collection of all q with $a, g, j \notin q$), we have $s|_V = f/g = h/j = t|_V \in \mathcal{O}_X(V)$. Therefore, $s = t \in \mathcal{O}_p$, so this map is injective.

Now, we prove (2).

We will define a map $\psi : A_f \to \mathcal{O}_X(U)$ by sending an element a/f^n to the section $s = a/f^n$ (this is an element of $\mathcal{O}_X(U)$ by definition; for any $p \in U$, $f \notin p$). To show this is injective, we must prove that if two elements a/f^n and b/f^m have the same value in $\mathcal{O}_X(U)$, which *by definition means they have the same value in each A_p for $p \in D(f)$*, then they are actually equal in A_f, meaning there is some f^l such that $f^l(af^m - bf^n) = 0$.

So, assume they have the same image, i.e. for every $p \in D(f)$, $a/f^n = b/f^m \in A_p$, so there is some $h \notin p$ with $h(af^m - bf^n) = 0$. For notational simplicity, let $g = af^m - bf^n$. Then, $h \in Ann(g)$ but $h \notin p$, so $Ann(g) \not\subset p$, and this holds for any $p \in D(f)$, so $V(Ann(g))$ does not contain any prime p in $D(f)$, i.e. $V(Ann(g) \cap D(f) = \emptyset$, which implies $V(Ann(g)) \subset V((f))$. This means $f \in \sqrt{Ann(g)}$, so $f^l \in Ann(g)$ for some $l > 0$, so $f^l(af^m - bf^n) = 0$, which implies that $a/f^n = b/f^m$ in A_f. Therefore, the map is injective.

Now, we must show that ψ is surjective. We will make several simplifying changes through the course of the proof, which I will label clearly.

Suppose $s \in \mathcal{O}_X(U)$. We want to show that s comes from some element in A_f, i.e. $s = a/f^n$ for some $a \in A$. By definition of $\mathcal{O}$, we can over U by open sets $U = \cup V_i$ where on each i, $s_i := s|_{V_i} = a_i/g_i$ where $a_i, g_i \in A$ and $g_i \notin p$ for all $p \in V_i$ (i.e. $V_i \subset D(g_i)$).

Simplification 1: Because the open sets of the form $D(h)$ form a base for the topology, by (possibly passing to an open cover of V_i) we may assume $V_i = D(h_i)$ for some $h_i \in A$.

Also, $V_i = D(h_i) \subset D(g_i)$, so $V((g_i)) \subset V((h_i))$, so $\sqrt{(h_i)} \subset \sqrt{(g_i)}$, so $h_i^n \in (g_i)$ for some $n > 0$. Therefore, for some $c \in A$, $h_i^n = cg_i$ so $a_i/g_i = ca_i/cg_i = ca_i/h_i^n$.

Simplification 2: Because $D(h_i) = D(h_i^n)$, we can replace h_i^n with h_i' and ca_i with a_i' to assume the following: U is covered by open sets $D(h_i')$, and $s_i := s|_{D(h_i')} = a_i'/h_i'$. Because we will simplify again, we will drop the primes from the notation, so we have: U is covered by open sets $D(h_i)$, and $s_i := s|_{D(h_i)} = a_i/h_i$

Simplification 3: We can assume the covering $D(h_i)$ is finite. Why? If $D(f) \subset \cup D(h_i)$, this means $V(\sum(h_i)) = \cap V((h_i)) \subset V((f))$, which means $f \in \sqrt{\sum(h_i)}$. In other words, $f^n \in \sum(h_i)$, but by definition of the sum of ideals, any element in $\sum(h_i)$ is a *finite* sum of elements, so $f^n = \sum_{i=1}^r b_i h_i$ for some $b_i \in A$, $r > 0$. Therefore, $D(f) \subset D(h_1) \cup \cdots \cup D(h_r)$.

Now, we wish to say that our element s is equal to a/f^n. Roughly, we will patch together the s_i's over common denominators to do this. Because s was a sheaf, the s_i's (restrictions of s) must agree on the overlap of the sets $D(h_i)$, so on $D(h_i) \cap D(h_j) = D(h_i h_j)$, $s_i = a_i/h_i = s_j = a_j/h_j$. By the injectivity that we have already proven, this means the elements have the same image in $\mathcal{O}(D(h_i h_j))$, so they are actually equal in $A_{h_i h_j}$. Therefore, for some n, $(h_i h_j)^n(a_i h_j - a_j h_i) = 0$. Because there are only finitely many intersections, we may choose $n \gg 0$ so it works for all i, j. In other words, we are assuming for any i, j, $h_j^{n+1} h_i^n a_i - h_i^{n+1} h_j^n a_j$.

Simplification 4: Because $D(h_i) = D(h_i^{n+1})$, we can replace our open cover by $D(h_i^{n+1})$ (and rename h_i^{n+1} as h_i', and $h_i^n a_i$ by a_i') to assume we have an open cover $D(f) \subset D(h_1') \cup \cdots \cup D(h_r')$ where $s_i = a_i'/h_i'$, and for all i, j, $a_i' h_j' = a_j' h_i'$. Again, we will drop all of the primes.

So, we have an open cover $D(f) \subset D(h_1) \cup \cdots \cup D(h_r)$ where $s_i = a_i/h_i$, and for all i, j, $a_i h_j = a_j h_i$.

Now, because $D(f) \subset D(h_1) \cup \cdots \cup D(h_r)$, there is some n such that $f^n = \sum_{i=1}^r b_i h_i$. Let $a = \sum b_i a_i$. Then, $h_j a = \sum_{i=1}^r b_i h_j a_i$, and we can use the relationship $a_i h_j = a_j g_i$ to write $h_j a = \sum_{i=1}^n b_i a_j h_i = a_j f^n$. This means $h_j a - a_j f^n = 0$, or $a/f^n = a_j/h_j$ on $D(h_j)$. In other words, the image of the element a/f^n in $\mathcal{O}_X(U)$ agrees with $s_i = s|_{D(h_i))}$ on the open cover. So, by the sheaf condition, the image of the element $a/f^n \in \mathcal{O}_X(U)$ is s. Because $a/f^n \in A_f$, this shows that ψ is surjective. $\square$