ALGEBRAIC GEOMETRY: MONDAY, MARCH 20

1. Introduction to Schemes

Definition 1.1. Let A be a ring. Define Spec A to be the set of prime ideals in A.

We will regard Spec A as a topological space with the following closed sets: for any ideal $I \subset A$, let

$$V(I) = \{P \in \text{Spec } A \mid I \subset P\}.$$

Some examples (some are repeated from last time):

Example 1.2. What is Spec k, where k is a field? The only ideals are (0) and (1), so the closed sets are the whole space and the empty set. This is just one point.

Example 1.3. Let R be a DVR (equivalently, an integrally closed Noetherian local ring of dimension 1), like $R = k[x]_{(x)}$. This has two prime ideals: (0) (because it is an integral domain), and m, the unique maximal ideal. So, Spec $R = \{(0), m\}$. Because m is maximal, $V(m) = \{m\}$, so the ideal m is a closed point. This means that (0) is open because it is the complement of $V(m)$ in Spec R, but $V((0)) = \text{Spec } R$ because (0) is contained in both ideals in Spec R. So, this space has two points, one of which is open and dense, and the other which is closed.

Example 1.4. What is Spec $\mathbb{R}[x]$? Certainly there is (0), but this is contained in every prime ideal, so this is a point whose closure is the whole space. This is called the generic point.

The other points are the other prime ideals. We definitely have maximal ideals, of the form $(x-a)$ for $a \in \mathbb{R}$. These are closed points (because they are maximal ideals), and are in bijection with $\mathbb{R}$.

Also, any irreducible polynomial $f(x)$ defines a prime ideal (and in fact maximal ideal). Irreducible polynomials are either linear, $x - a$, which we have already discussed, or irreducible quadratics. For example, $f(x) = x^2 + 1$. These are also closed points, and are in bijection with points in the upper half plane (identifying them with the complex root with positive imaginary part). So, you could say Spec $\mathbb{R}(x)$ 'looks like' the upper half plane together with the real line.

Reminder from last time: if $k = \overline{k}$, then Spec $k[x]$ has only the maximal ideals, so the closed points just look like k (or, $\mathbb{A}^1_k$), and then we have the 'extra' generic point that is dense (so we imagine it living generically at every point).

Definition 1.5. The affine line $\mathbb{A}^1_k$ is the topological space Spec $k[x]$.

Example 1.6. What is Spec $k[x, y]$, where k is algebraically closed? The prime ideals are: (0), $\{(x - a, y - b) \mid a, b \in k\}$, and $\{(f) \mid f \in k[x, y] \text{ irreducible }\}$. The maximal ideals $\{(x - a, y - b) \mid a, b \in k\}$ correspond to $(a, b) \in k^2$, so look like our earlier definition of the affine plane. We also have the generic point (0) whose closure is the whole space, so it is a dense point that lives 'everywhere.' Finally, an ideal of the last type is a 'generic point' on the curve $f(x, y) = 0$, because its closure contains every maximal ideal defining a point on the curve.

Armed with these examples, we move to define *affine schemes.* Affine schemes will consist of two pieces of data: (1) the topological space Spec A, and (2) the structure sheaf of Spec A, defined below.

Definition 1.7. Suppose $X = \text{Spec } A$. The **structure sheaf** of X is the sheaf $\mathcal{O}_X$ defined as follows. For $p \in \text{Spec } A$, let A_p be the localization of A at the prime p. For any open set U, let $\mathcal{O}_X(U)$ be the set of functions $s : U \to \cup_{p \in U} A_p$ such that in a neighborhood V of each $p \in U$, $s|_V = \frac{f}{g}$, where $f, g \in A$, $g \notin q$ for any $q \in V$.

In other words, we are saying that the 'regular functions' on Spec A are locally quotients f/g of elements of A, and 'the denominator being nonzero' at a point $p \in \text{Spec } A$ means that $g \notin p$ so that $f/g \in A_p$.

Exercise 1.8. Verify that $\mathcal{O}_X$ is a sheaf.

Now, some properties of $\mathcal{O}_X$. *Notation: if $f \in A$, let $D(f)$ be the complement of $V((f))$.*

Lemma 1.9. *The open sets $D(f)$ form a base for the topology on* Spec A*.*

Proof. By definition, $D(f)$ is an open set. We need to verify that, for any open $U \subset \text{Spec } A$ and point $p \in U$ there exists an open set $D(f)$ such that $p \in D(f) \subset U$.

Let $V(I)$ be the complement of U. Because $p \in U$, $p \notin V(I)$ (and remember that $p \in \text{Spec } A$ is a prime ideal of A), so by definition of $V(I)$, I is not contained in p. Therefore, there exists some $f \in I$, $f \notin p$, and $(f) \subset I$, so $V(I) \subset V((f))$, so $D(f) \subset U$. Also, $D(f)$ contains p by definition. □

What does $\mathcal{O}_X$ actually look like? The following proposition will tell us.

Proposition 1.10. *Let $X = \text{Spec } A$ and $\mathcal{O}_X$ be the structure sheaf.*

(1) For any point $p \in \text{Spec } A$ (= prime ideal of A), the stalk of $\mathcal{O}_X$ at p is $\mathcal{O}_p \cong A_p$.
(2) If $U = D(f)$, $f \in A$, is an open set in the base for the topology, then $\mathcal{O}_X(U) \cong A_f$. In particular, $X = D(1)$, so $\mathcal{O}_X(X) \cong A$.

The proof will be next time, but first some notation/reminder: if S is a multiplicatively closed subset of A, then

$$S^{-1}A = \{\frac{a}{s} \mid a \in A, s \in S\}/\sim$$

where two elements are equivalent if:

$$a/s \sim b/t \iff \text{there exists } r \in S \text{ such that } r(at - bs) = 0.$$

If p is a prime ideal in A, let $S_p = \{a \in A \mid a \notin p\}$ and define $A_p := S_p^{-1}A$ (we invert the elements of A outside of p). If f is an element of A, then let $S_f = \{1, f, f^2, \dots\}$ and define $A_f := S_f^{-1}A$.