ALGEBRAIC GEOMETRY: FRIDAY, MARCH 10

1. Sheaves, continued.

A few more definitions:

Definition 1.1. A **subsheaf** of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that, for every $U \subset X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps on $\mathcal{F}'$ are induced by those of $\mathcal{F}$. It follows that $\mathcal{F}'_p$ is a subgroup of $\mathcal{F}_p$.

Definition 1.2. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the **kernel** of ϕ, $\ker \phi$, is the presheaf kernel of ϕ (which is a sheaf). This is a subsheaf of $\mathcal{F}$.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is **injective** if $\ker \phi = 0$.

Definition 1.3. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, the **image** of ϕ, $\text{im}\phi$, is the sheafification of the presheaf image of ϕ. By the universal property of sheafification, this can be identified as a subgroup of $\mathcal{G}$.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is **surjective** if $\text{im}\phi = \mathcal{G}$.

Definition 1.4. An **exact sequence of sheaves** is a sequence of sheaves

$$\to \mathcal{F}^{i-1} \to \mathcal{F}^{i} \to \mathcal{F}^{i+1} \to$$

such that, at each stage, $\ker \phi^i = \text{im}\phi^{i-1}$.

Definition 1.5. If $\mathcal{F}' \subset \mathcal{F}$, the **quotient sheaf** $\mathcal{F}/\mathcal{F}'$ is the sheafification of the presheaf whose values on each open set are $\mathcal{F}(U)/\mathcal{F}'(U)$.

Definition 1.6. The **cokernel** of a map of sheaves $\mathcal{F} \to \mathcal{G}$ is the sheafification of the presheaf cokernel.

Now, suppose $f : X \to Y$ is a continuous map of topological spaces.

Definition 1.7. If $\mathcal{F}$ is a sheaf on X, the **direct image** or **pushforward** of $\mathcal{F}$ is the sheaf $f_*\mathcal{F}$ such that $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$.

If $\mathcal{G}$ is a sheaf on Y, the **inverse image** of $\mathcal{G}$ is the sheaf $f^{-1}\mathcal{G}$ that is the sheafification of the presheaf whose values on any open set U are $f^{-1}\mathcal{G}(V) = \lim \mathcal{G}(V)$, where the limit is taken over all open sets V containing $f(U)$.

If $Z \subset X$, let $i : Z \to X$ denote the inclusion. If $\mathcal{F}$ is a sheaf on X, we denote $i^{-1}\mathcal{F}$ by $\mathcal{F}|_Z$ and call it the **restriction** of $\mathcal{F}$. You can verify that, for $p \in Z$, $(\mathcal{F}|_Z)_p = \mathcal{F}_p$.

Exercise 1.8. Let $f \in A$ be an irreducible polynomial and let $X = Z(f) \subset \mathbb{A}^n$. Let $\mathcal{I}_X(U) = \{f \in \mathcal{O}_{\mathbb{A}^n}(U) \mid f(p) = 0 \quad \forall p \in X \cap U\}$. Show that $\mathcal{I}_X$ is a sheaf, a subsheaf of $\mathcal{O}_{\mathbb{A}^n}$, and that the following sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{A}^n} \to i_*\mathcal{O}_X \to 0$$

is exact, where $i : X \to \mathbb{A}^n$ is the inclusion.

2. Spec

As a first step to defining schemes, we start with the spectrum of a ring.

Definition 2.1. Let A be a ring. Define Spec A to be the set of prime ideals in A.

We will regard Spec A as a topological space. For any ideal $I \subset A$, let

$$V(I) = \{P \in \text{Spec } A \mid I \subset P\}.$$

This will give the topology after the following lemma.

Lemma 2.2. *(1) If I_1, I_2 are ideals of A, then $V(I_1 I_2) = V(I_1) \cup V(I_2)$.*

(2) If $\{I_i\}$ is a set of ideals in A, then $V(\sum I_i) = \cap V(I_i)$.

(3) If I_1 and I_2 are two ideals of A, then $V(I_1) \subset V(I_2)$ if and only if $\sqrt{I_2} \subset \sqrt{I_1}$.

Proof. For (1), suppose P contains I_1 or P contains I_2. Then, $I_1 I_2 \subset P$. Now suppose $I_1 I_2 \subset P$. If $I_2 \not\subset P$, then there exists $r \in I_2$ such that $r \notin P$, so $I_1 r \subset P$ implies that $I_1 \in P$.

For (2), recall that $\sum I_i$ is the smallest ideal containing all of the ideals I_i. So, I_i is contained in P for each i if and only if $\sum I_i$ is contained in P.

(3) follows because the radical of an ideal I is the intersection of all prime ideals containing I. □

We define the Zariski topology on Spec A to be the topology where the closed sets are $V(I)$ for any ideal $I \subset A$.

Let's do some examples!

Example 2.3. What is Spec k, where k is a field? The only ideals are (0) and (1), so the closed sets are the whole space and the empty set. This is just one point.

Example 2.4. What is Spec $k[x]$? Certainly there is (0), but this is contained in every prime ideal, so this is a point whose closure is the whole space. This is called the generic point.

The other points are the other prime ideals. If k is algebraically closed, all of the other prime ideals are maximal, of the form $(x - a)$ for $a \in k$. These are closed points (because they are maximal ideals), and are in bijection with k. So, in a sense we will get to, this is $\mathbb{A}^1_k$.

What if k is not algebraically closed? Suppose $k = \mathbb{R}$. Then, any irreducible polynomial $f(x)$ defines a prime ideal (and in fact maximal ideal). Irreducible polynomials are either linear, $x - a$, which correspond to the values in $\mathbb{R}$, or irreducible quadratics. For example $x^2 + 1$. These are also closed points!

Exercise 2.5. What is Spec $\mathbb{Z}$?

Exercise 2.6. What is Spec $k[x]/(x^2)$?

Exercise 2.7. What is Spec $k[x, y]$?