ALGEBRAIC GEOMETRY: WEDNESDAY, MARCH 8

1. Sheaves, continued.

Reminders from last time:

Definition 1.1. Let X be a topological space. A **presheaf $\mathcal{F}$ of abelian groups on** X is the data of:

(1) For every open set $U \subset X$, an abelian group $\mathcal{F}(U)$, such that $\mathcal{F}(\emptyset) = 0$
(2) For every $V \subset U$, there is a $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_{UU} = id$ and if $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

A sheaf is presheaf that is defined by 'local data':

Definition 1.2. Let X be a topological space. A **sheaf $\mathcal{F}$ on** X is a presheaf $\mathcal{F}$ satisfying the following additional conditions:

(1) For any open set $U \subset X$ and open cover $U = \cup V_\alpha$, if $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha}(s) = 0$ for each α, then $s = 0$. (If s 'restricts' to 0 on each open set, then is is 0.)
(2) For any open U and open cover $U = \cup V_\alpha$, if for each α there exists $s_\alpha \in \mathcal{F}(V_\alpha)$ such that $\rho_{V_\alpha(V\alpha \cap V_\beta)} s_\alpha = \rho_{V_\beta(V\alpha \cap V_\beta)} s_\beta$, then there exists $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha})(s) = s_\alpha$. (If there exists a collection of s's on the open cover that agree on the overlaps, they can be 'glued' together to an s on the whole set.)

Definition 1.3. If $\mathcal{F}$ is a presheaf on X and $p \in X$, then the **stalk** of $\mathcal{F}$ at p is the direct limit over sets containing p:

$$\mathcal{F}_p = \lim_{p \in U} \mathcal{F}(U)$$

where the limit is taken over the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$.

Definition 1.4. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves (or sheaves) on a topological space X, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subset X$ such that, if $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\rho_{UV}} & & \downarrow{\rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

commutes. By abuse of notation, ρ means the restriction map for either $\mathcal{F}$ or $\mathcal{G}$.

Definition 1.5. An **isomorphism** of presheaves (or sheaves) is a morphism with a two-sided inverse.

Proposition 1.6. *If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then ϕ is an isomorphism if and only if the induced map $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every $p \in X$.*

Proof. If ϕ is an isomorphism, then by definition ϕ_p is an isomorphism. So, we assume $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism such that ϕ_p is an isomorphism for each p. We will use the local nature of sheaves to show that $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for each U (which will show that ϕ is an isomorphism).

First we show $\phi(U)$ is injective. If $s \in \mathcal{F}(U)$ such that $\phi(s) = 0$, then for each $p \in U$, $\phi_p(s) = 0$, so $s_p = 0$ (because ϕ_p was an isomorphism). By definition of stalk, there is a neighborhood V_p containing p such that $s|_{V_p} = 0$. Because these neighborhoods cover X, by the first sheaf property, $s = 0$.

Now, we show $\phi(U)$ is surjective. If $t \in \mathcal{G}(U)$, consider $t_p \in \mathcal{G}_p$ for any $p \in U$. Because ϕ_p is an isomorphism, there is an element $s_p \in \mathcal{F}_p$ such that $\phi_p(s_p) = t_p$. By definition of stalk, there is a neighborhood V_p of p such that s_p is defined by $s(p)$ an element of $\mathcal{F}(V_p)$. Then, $\phi(s(p))$ agrees with $t|_{V_p}$ on some (possibly smaller) neighborhood of p. On this smaller neighborhood, we can say that $\phi(s(p)) = t|_{V_p} \in \mathcal{G}(V_p)$. Now, because U is covered by the open sets V_p, we want to patch the local functions together $s(p)$ to a function on U. To do this, we need to show the local functions agree on the overlaps. If $p, q \in X$, then $s(p)|_{V_p \cap V_q} = s(q)|_{V_p \cap V_q}$ because $\phi(s(p)|_{V_p \cap V_q}) = t|_{V_p \cap V_q} = \phi(s(q)|_{V_p \cap V_q})$ and ϕ is injective on open sets. Therefore, they agree on the overlap so by the second sheaf property, there exist an element $s \in \mathcal{F}(U)$ such that $s|_{V_p} = s(p)$. Furthermore, because $\phi(s|_{V_p}) - t|_{V_p} = 0$ by construction, $\phi(s) - t = 0$ in a neighborhood of each point, so by the first sheaf property, $\phi(s) - t = 0$, i.e. $\phi(s) = t$. Therefore, $\phi(U)$ is surjective. □

Definition 1.7. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, the

1. **presheaf kernel** is the presheaf whose values on each open set is $\ker \phi(U)$
2. **presheaf cokernel** is the presheaf whose values on each open set is $\mathrm{coker}\phi(U)$
3. **presheaf image** is the presheaf whose values on each open set is $\mathrm{im}\phi(U)$

Exercise 1.8. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, the presheaf kernel of ϕ is a sheaf, but the presheaf cokernel and presheaf image need not be sheaves.

Example 1.9. To see a concrete example, let $X = \mathbb{C} - 0$ and let $\mathcal{F} = \mathcal{G}$ be the sheaf of continuous functions on X. Consider the map $\phi : \mathcal{F} \to \mathcal{G}$ given by $\phi(f) = e^f$.

The image of this map is *not a sheaf.* For example, the function z (the identity) has a logarithm locally at every point, i.e. there is an open cover $X = \cup U_i$ such that $z = \log f_i$, so $z = \phi(f_i)$ on U_i, but there is no continuous logarithm on all of X, so these functions do not come from an element of the image of ϕ. Therefore, it fails the second property of being a sheaf.

Definition 1.10. Given a presheaf, we can define the **sheafification** of $\mathcal{F}$ to the unique sheaf $\mathcal{F}^{sh}$ with morphism $\theta : \mathcal{F} \to \mathcal{F}^{sh}$ such that, for any morphism $\phi : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$, the morphism ϕ factors through θ.

We verify that $\mathcal{F}^{sh}$ exists. (The uniqueness of $\mathcal{F}^{sh}$ follows from the universal property.) To construct $\mathcal{F}^{sh}$, we define it on open sets. Essentially, we will just declare that all compatible germs of functions form a section:

$$\mathcal{F}^{sh}(U) = \{(f_p) \in \cup_{p \in U} \mathcal{F}_p \mid \text{ for all } p \in U, \text{ there exists } V \subset U \text{ containing } p \text{ and } s \in \mathcal{F}(V) \text{ such that, for all } q \in V, s_q = f_q\}$$

Exercise 1.11. Verify that $\mathcal{F}^{sh}$ is actually a sheaf and that it has the universal property as above. Show that $\mathcal{F}_p = \mathcal{F}^{sh}_p$, and if $\mathcal{F}$ was already a sheaf, then $\mathcal{F} = \mathcal{F}^{sh}$.

Definition 1.12. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the **kernel** of ϕ, $\ker \phi$, is the presheaf kernel of ϕ (which is a sheaf).

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is **injective** if $\ker \phi = 0$ (equivalently, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every U).

Definition 1.13. If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, the **image** of ϕ, $\text{im}\phi$, is the sheafification of the presheaf image of ϕ.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is **surjective** if $\text{im}\phi = \mathcal{G}$.

Caution: unlike the injective case, a surjective morphism of sheaves need not induce a surjection of sections $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$. (This is true only on the level of stalks!)

Example 1.14. Consider the previous example of the function e^f. The sheafification of the image must include the function z (the identity) but this is not in the image of $\mathcal{F}(X)$. Therefore, it is not surjective on global sections.

Definition 1.15. The **cokernel** of a map of sheaves $\mathcal{F} \to \mathcal{G}$ is the sheafification of the presheaf cokernel.