ALGEBRAIC GEOMETRY: MONDAY, MARCH 6

1. Sheaves

We recall the definition of a sheaf from earlier.

Definition 1.1. Let X be a topological space. A **presheaf** $\mathcal{F}$ **of abelian groups on** X is the data of:

(1) For every open set $U \subset X$, an abelian group $\mathcal{F}(U)$, such that $\mathcal{F}(\emptyset) = 0$
(2) For every $V \subset U$, there is a $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_{UU} = id$ and if $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

We can replace 'abelian groups' by other categories (rings, sets, etc) and define presheaves with values in other categories.

A sheaf is presheaf that is defined by 'local data':

Definition 1.2. Let X be a topological space. A **sheaf** $\mathcal{F}$ **on** X is a presheaf $\mathcal{F}$ satisfying the following additional conditions:

(1) For any open set $U \subset X$ and open cover $U = \cup V_\alpha$, if $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha}(s) = 0$ for each α, then $s = 0$. (If s 'restricts' to 0 on each open set, then is is 0.)
(2) For any open U and open cover $U = \cup V_\alpha$, if for each α there exists $s_\alpha \in \mathcal{F}(V_\alpha)$ such that $\rho_{V_\alpha(V\alpha \cap V_\beta)} s_\alpha = \rho_{V_\beta(V\alpha \cap V_\beta)} s_\beta$, then there exists $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha})(s) = s_\alpha$. (If there exists a collection of s's on the open cover that agree on the overlaps, they can be 'glued' together to an s on the whole set.)

Definition 1.3. If $\mathcal{F}$ is a sheaf on X and $U \subset X$ is an open set, an element $s \in \mathcal{F}(U)$ is called a **section** of $\mathcal{F}$ over U. An element $s \in \mathcal{F}(X)$ is called a **global section** of $\mathcal{F}$.

The maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are called restriction maps, and the image of $s \in \mathcal{F}(U)$ is often denoted by $s|_V$.

Definition 1.4. If X is a variety, the sheaf of rings $\mathcal{O}_X$ is called the **structure sheaf** of X.

Example 1.5. If G is an abelian group and X a topological space, the **constant sheaf** $\mathcal{G}$ is the sheaf whose values on connected open sets are just elements of G. To be precise, give G the discrete topology, and let $\mathcal{G}(U) = \{$ continuous maps $U \to G\}$.

So, if U is connected, $\mathcal{G}(U) = G$, and if U is a union of (open) connected components, $\mathcal{G}(U)$ is a direct sum of copies of G.

We could also define the **constant presheaf** $\mathcal{G}'$ to be $\mathcal{G}'(U) = G$ for any nonempty open set U. This is not a sheaf! For example, if we take $G = \mathbb{Z}_2$ and suppose U is disconnected, $U = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$, then we could have the element $1 \in \mathcal{G}'(V_1)$ and $0 \in \mathcal{G}'(V_2)$ (a constant function on each open set, but a different value on each one). There is no global element in $\mathcal{G}'(U)$ that restricts to be 1 on one component and 0 on the other, so this violates the second sheaf condition!

Definition 1.6. If $\mathcal{F}$ is a presheaf on X and $p \in X$, then the **stalk** of $\mathcal{F}$ at p is the direct limit over sets containing p:

$$\mathcal{F}_p = \lim_{p \in U} \mathcal{F}(U)$$

where the limit is taken over the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$.

By definition of direct limit, the elements of $\mathcal{F}_p$ are *germs* of $\mathcal{F}$ at p, i.e. elements $s \in \mathcal{F}(U)$ where $p \in U$, and two elements $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are equivalent if they agree on some $W \subset U \cap V$: $s|_W = t|_V$.

Definition 1.7. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves (or sheaves) on a topological space X, a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subset X$ such that, if $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi(V)} & \mathcal{G}(V) \end{array}$$

commutes. By abuse of notation, ρ means the restriction map for either $\mathcal{F}$ or $\mathcal{G}$.

Definition 1.8. An **isomorphism** of presheaves (or sheaves) is a morphism with a two-sided inverse.