ALGEBRAIC GEOMETRY: FRIDAY, FEBRUARY 24

1. Rational Maps and Blow-ups, Continued

Reminders from last time:

Definition 1.1. Let X and Y be varieties. A **rational map** $\phi : X \dashrightarrow Y$ is an equivalence class of pairs (U, ϕ_U), where $U \subset X$ is an open set and $\phi_U : U \to Y$ is a morphism. Two pairs (U, ϕ_U) and (V, ϕ_V) are equivalent if $\phi_U = \phi_V$ on $U \cap V$.

Definition 1.2. A rational map $\phi : X \dashrightarrow Y$ is **dominant** if the image of U is dense in Y.

Definition 1.3. Two varieties X and Y are said to be **birational** if there is a rational map $\phi : X \dashrightarrow Y$ which admits an inverse $\psi : Y \dashrightarrow X$.

Example 1.4. Let $p = (0, \dots, 0) \subset \mathbb{A}^n$. The **blow-up** of the point $p \in \mathbb{A}^n$ is the quasi-projective variety $Y \subset \mathbb{A}^n_{x_i} \times \mathbb{P}^{n-1}_{y_i}$ defined by the equations

$$\{x_i y_j - x_j y_i = 0 \mid i, j = 1, \dots, n\}.$$

Let $pr_1 : \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ be the projection onto the first factor, and let $\phi = pr_1|_Y : Y \to \mathbb{A}^n$. This is a birational map! Indeed, consider the set $U = \mathbb{A}^n - p$. We claim that $Y \cap \phi^{-1}(U) \cong U$. Suppose $q = (a_1, \dots, a_n) \neq 0$. Then, at least one $a_i \neq 0$, so $\phi^{-1}(q)$ is one point: we must have $y_j = (a_j/a_i)y_i$ (and we can scale y_i by any constant, so suppose $y_i = a_i$), which says $\phi^{-1}(q) = q \times [a_1 : \cdots : a_n]$. This says the morphism $\psi : U \to Y \cap \phi^{-1}(U)$ given by $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n) \times [a_1 : \cdots : a_n]$ is the inverse of ϕ, so ϕ is birational.

What does this morphism do over p? The points of $\phi^{-1}(p)$ exactly correspond to lines through $p \in \mathbb{A}^n$: if L is a line through p, then L consists of all points $(x_1, \dots, x_n) = (ta_1, \dots, ta_n)$ for some point $q = (a_1, \dots, a_n)$ and $t \in \mathbb{A}^1$. Then, if we consider $L - p$ (by what we just showed), ϕ has an inverse ψ, and the preimage is given (parametrically) by $(ta_1, \dots, ta_n) \times [ta_1 : \cdots : ta_n]$ for $t \neq 0$. But, the second point is in projective space, so this is the same as $(ta_1, \dots, ta_n) \times [a_1 : \cdots : a_n]$. As we approach 0, this shows that the closure of the line $\phi^{-1}(L - p)$ is the line that meets $\phi^{-1}(p)$ in just the point $[a_1 : \cdots : a_n]$. This gives a one-to-one correspondence between points of the $\mathbb{P}^{n-1}$ living over p and lines through p.

Example 1.5. If $V \subset \mathbb{A}^n$ is an affine variety containing p, then we can define the blow-up of V at p to be the closure of $\phi^{-1}(V - p)$ inside the blow-up of $\mathbb{A}^n$ at p.

For example, suppose $V = Z(y^2 - x^2(x + 1)) \subset \mathbb{A}^2$. This is a nodal curve, and let's blow up the node $p = (0, 0) \in \mathbb{A}^2$. Let's call the coordinates on the $\mathbb{P}^1$ u and v, so the blow-up of $\mathbb{A}^2$ is given by $Z(xv - yu) \subset \mathbb{A}^2 \times \mathbb{P}^1$.

To compute the blow up of V, we need to determine the closure of the preimage of $V - p$. We will work affine-locally, meaning we first look on the $\mathbb{P}^1$-chart where $u \neq 0$. This is just $D(u) \cong \mathbb{A}^1$, and we will call the coordinate on $\mathbb{A}^1$ v. On $\mathbb{A}^2 \times D(u)$, the blow-up is given by $xv - y = 0$ (we 'set u equal to 1' in the isomorphism $D(u) \cong \mathbb{A}^1$), which says $y = xv$. Taking the equation of V and substituting, we get $(xv)^2 - x^2(x+1) = x^2(v^2 - (x+1))$. If q is any point in $V - p$, then $x \neq 0$, so the equation of $V - p$ in the blow-up is given by $v^2 - (x + 1) = 0$. The closure of this in $\mathbb{A}^2 \times D(u)$ is just $Z(v^2 - (x + 1))$, which, by definition, is the blow-up of V on this chart! As an exercise, you can work out the equation on the other chart.

Observe what happens over the point p. This equation intersects the pre-image of p (where $(x, y) = (0, 0)$) is $v^2 - (0+1) = 0$, or $v^2 = 1$, so $v = \pm 1$. This says the blow-up of V intersects the preimage of $(0,0)$ in *two* distinct points, $[1:1]$ and $[1:-1]$. So, what we have accomplished by blowing-up is separating the two branches of the curve V at the node. This is called a 'resolution of singularities' because the blow-up of V is smooth everywhere.

To make the previous example more precise (and to generalize it), we will introduce what it means for a variety to be smooth.

2. Nonsingularity

Definition 2.1. If $Y \subset \mathbb{A}^n$ is an affine variety with ideal $I(Y) = (f_1, \ldots, f_t) \subset k[x_1, \ldots, x_n]$, the Y is **nonsingular** or **smooth** at a point $p \in Y$ if the rank of the Jacobian matrix (the matrix whose ijth entry is $\partial f_i/\partial x_j$) at p is $n - \dim Y$. It is **singular** at p if it is not nonsingular at p. It is **nonsingular** if it is nonsingular at every point.

Example 2.2. The curve given by $y^2 - x^2(x+1) = 0$ in $\mathbb{A}^2$ is singular at $(0,0)$, and nonsingular at every other point. The Jacobian matrix is $[-3x^2 - 2x \quad 2y]$ which has rank 1 if and only if $2y \neq 0$ and $-3x^2 - 2x = -x(3x+2) \neq 0$, which occurs if and only if $(x,y) \neq (0,0), (-2/3, 0)$, but the latter point is not on the curve, so the only singularity is $(0,0)$.

While the definition appears to depend on the embedding of Y into affine space, it turns out it is intrinsic to Y.

Definition 2.3. If A is a noetherian local ring with maximal ideal m and residue field k, then A is **regular** if $\dim_k m/m^2 = \dim A$.

Theorem 2.4. *If $Y \subset \mathbb{A}^n$ is an affine variety and $p \in Y$, then Y is nonsingular at p if and only if $\mathcal{O}_p$ is a regular local ring.*

Proof. Let $I(Y) = (f_1, \ldots, f_t)$ and J be the Jacobian matrix. Let m be the maximal ideal of $\mathcal{O}_p$. To prove this, we will show that $\dim m/m^2 + \dim J = n$, which says exactly that $\dim J = n - \dim m/m^2$.

This will prove the result: we know already that $\dim \mathcal{O}_p = \dim Y$, so $\mathcal{O}_p$ is regular if and only if $\dim m/m^2 = \dim Y$ if and only if $\dim J = n - \dim Y$.

So, let's show $\dim m/m^2 + \dim J = n$. Let $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ and let

$$m_p = (x_1 - a_1, \ldots, x_n - a_n) \subset A = k[x_1, \ldots, x_n]$$

be the corresponding maximal ideal. Let

$$\theta : A \to k^n$$

be given by

$$\theta(f) = (\partial f/\partial x_1(p), \ldots, \partial f/\partial x_n(p)).$$

The vectors $\theta(x_i - a_i)$ form a basis for k^n and $\theta(m_p^2) = 0$, so θ gives an isomorphism

$$\overline{\theta} : m_p/m_p^2 \to k^n.$$

For simplicity, let $V = m_p/m_p^2$.

By construction, $\text{rank} J$ is $\dim \theta(I(Y)) \subset k^n$. Let $\overline{I}$ be the image of $I(Y)$ in V, which is isomorphic to $\theta(I(Y))$. Note that $\overline{I}$ is just $\overline{I} = (I(Y) + m_p^2)/m_p^2$. Because V is a vector space, $V = V/\overline{I} \oplus \overline{I}$, and $\dim V = n$, and $\dim \overline{I} = \text{rank} J$. So, we just need to show that $\dim V/\overline{I} = \dim m/m^2$.

But, $\mathcal{O}_p$ is just $(A/I)_{m_p}$, and m is just the maximal ideal which is the image of m_p. In other words, $m = (m_p)/I$, so $m^2 = (m_p^2 + I)/I$, so $m/m^2 = (m_p)/(m_p^2 + I)$, which is just $V/\overline{I}$, so $\dim V/\overline{I} = \dim m/m^2$. □

We use this definition to generalize nonsingularity to all varieties. However, because the local ring $\mathcal{O}_p$ is something we can compute locally (i.e. on affine charts), we usually go to an affine chart and compute with the Jacobian condition there.

Definition 2.5. A variety Y is **nonsingular** at $p \in Y$ is $\mathcal{O}_p$ is a regular local ring, and **nonsingular** if it is nonsingular at every point.

Some remarks that we won't prove:

(1) If Y is a variety, then the set of singular points of Y is a proper closed subset of Y.
(2) Over $\mathbb{C}$, *Hironaka's resolution of singularities* says that, for any variety X, there is a nonsingular variety Y and a birational morphism $\phi : Y \to X$ (that is a composition of blow-ups!) such that $Y - \phi^{-1}\text{Sing}(X) \cong X - \text{Sing}(X)$. So, one reason we care about blow-ups is that they remove singular points of varieties!