ALGEBRAIC GEOMETRY: WEDNESDAY, FEBRUARY 22

1. Structure sheaves of projective varieties

Recall from last time: Y variety, $p \in Y$, $U \subset Y$ open.

Definition 1.1. Let $\mathcal{O}(U) = \{\text{regular functions on } U\}$ be the **structure sheaf** of Y.

Definition 1.2. The **local ring** of $p \in Y$ is

$$\mathcal{O}_{Y,p} = \{\text{germs of regular functions near } p\}.$$

This is defined with an equivalence relation: elements are pairs (U, f) where U is an open set and f is a regular function at p. If U, V are open sets containing p, $(U, f) = (V, g)$ if $f = g$ on $U \cap V$.

Definition 1.3. Given a variety Y, the **function field** of Y is

$$K(Y) = \{(U, f) \mid U \subset Y \text{ open }, f \in \mathcal{O}(U)\}/\sim$$

where $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$.

The elements of $K(Y)$ are **rational functions** on Y.

Last time we proved the following about affine varieties:

Theorem 1.4. *If $Y \subset \mathbb{A}^n$ is an affine variety, let $A(Y)$ be its affine coordinate ring. Then:*

1. $\mathcal{O}(Y) \cong A(Y)$
2. *For each $p \in Y$, define $m_p = \{f \in A(Y) \mid f(p) = 0\}$. The correspondence $p \mapsto m_p$ is a one-to-one correspondence between points of Y and maximal ideals in $A(Y)$.*
3. *For each $p \in Y$, $\mathcal{O}_p \cong A(Y)_{m_p}$ and $\dim \mathcal{O}_p = \dim Y$.*
4. *The field $K(Y)$ is isomorphic to the fraction field of $A(Y)$ (so, $K(Y)$ is a finitely generated field extension of k and $\mathrm{trdeg}_k K(Y) = \dim Y$.*

Today, the analogous statement for projective varieties.

NOTATION. Unfortunately, the notation regarding graded rings in this section is not great. I'll spell it out here:

1. If R is a graded ring and d a non-negative integer, then R_d is the degree d part of R.
2. If R is a ring and $f \in R$, then R_f is the localization of R at f, i.e. we can invert f (so we allow $f, f^2, f^3, \dots$ in the denominators).
3. If R is a graded ring and $f \in R$, then $R_{(f)}$ is the degree 0 part of R_f .
4. If R is a ring and p is a prime ideal, then R_p is the localization of R at the prime ideal p, which means everything *outside* of p is inverted. This is different than (2) and (3), and can be confusing but is supposed to be clear from context! For example, the notation $R_{(x)}$ could mean either the localization of R at the prime ideal (x) (where everything outside x is inverted) or it could mean the degree 0 part of R_x (the localization where x is inverted). If instead we write $R_{((x))}$ it must mean the degree 0 part of the localization of $R_{(x)}$.

Theorem 1.5. *Let $Y \subset \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$. Then:*

1. $\mathcal{O}(Y) = k$

(2) $\mathcal{O}_p = S(Y)_{(m_p)}$*, where* $m_p = \{f \in S(Y) \mid f(p) = 0\}$ *(from above:* $S(Y)_{(m_p)}$ *denotes the degree 0 elements in the localization* $S(Y)_{m_p}$*)*

(3) $K(Y) \cong S(Y)_{((0))}$ *(from above:* $S(Y)_{((0))}$ *denotes the degree 0 part of the fraction field of* $S(Y)$*)*

Proof. Let $U_i = D(x_i) \subset \mathbb{P}^n$, which is isomorphic to $\mathbb{A}^n$. Note the isomorphism and corresponding maps we defined earlier naturally gives an isomorphism of rings:

$$A = k[y_1, \dots, y_n] \to S_{(x_i)} = k[x_0, \dots, x_n]_{(x_i)}$$

(where $S_{(x_i)}$ is the degree 0 part of S_{x_i}) by

$$f(y_1, \dots, y_n) \mapsto f(x_0/x_i, \dots, \widehat{x_i/x_i}, \dots, x_n/x_i).$$

Let $Y_i = Y \cap U_i$ which is an affine variety. Then, one can check that this isomorphism of rings sends $I(Y_i)$ to $I(Y)S_{(x_i)}$, so we get an isomorphism $A(Y_i) \cong S(Y)_{(x_i)}$.

Now, we can essentially work with Y as if it were an affine variety because regular functions are defined locally. Let $p \in Y$, and choose i such that $p \in D(x_i)$. Then, by the theorem for affine varieties, $\mathcal{O}_p \cong A(Y_i)_{m_p}$, and because $x_i \notin m_p$, it is inverted when we localize at m_p, so the isomorphism above says $\mathcal{O}_p \cong A(Y_i)_{m_p} \cong S(Y)_{(m_p)}$.

Because $K(Y)$ is the quotient field of $\mathcal{O}_p$, $K(Y) \cong K(Y_i)$, so $K(Y) \cong S(Y)_{((0))}$.

Finally, if $f \in \mathcal{O}(Y)$, then f is regular on each Y_i, so $f \in A(Y_i) \cong S(Y)_{(x_i)}$, so on each Y_i, $f = g_i/x_i^{k_i}$ for some integer m with g_i homogeneous of degree k_i. Now, we want to show that this implies that f is actually constant. We know $x_i^{k_i} f \in S(Y)_{k_i}$ (the degree k_i part). Choosing $N \gg 0$, this implies $S(Y)_N \cdot f \subset S(Y)_N$, and even that $S(Y)_N \cdot f^q \subset S(Y)_N$ for all $q > 0$. In particular, it implies that, for any $q > 0$, $x_0^N f^q \in S(Y)$, so $S(Y)[f] \subset x_0^{-N} S(Y)$, so $S(Y)[f]$ is a finitely generated $S(Y)$-module. This implies that f is integral over $S(Y)$, i.e. there exist $a_i \in S$ such that (for some $m > 0$)

$$f^m + a_1 f^{m-1} + \dots a_m = 0.$$

As f was degree 0 and this holds for each graded piece, we may assume $a_i \in S(Y)$ has degree 0, i.e. $a_i \in k$, so f is actually algebraic over k. Therefore, because k is algebraically closed, $f \in k$. □

Before we move on to sheaves and schemes more generally, we will discuss two fundamental things about varieties that will appear often (e.g. in the learning seminar): birational maps and nonsingularity. We will revisit them in more detail in the future!

2. Rational Maps and Blow-ups

We have shown already that every open set of an irreducible topological space is dense, so encodes much of the information we want. With that motivation, in algebraic geometry, we will often consider morphisms that are not defined everywhere, but just on an open set.

Definition 2.1. Let X and Y be varieties. A **rational map** $\phi : X \dashrightarrow Y$ is an equivalence class of pairs (U, ϕ_U), where $U \subset X$ is an open set and $\phi_U : U \to Y$ is a morphism. Two pairs (U, ϕ_U) and (V, ϕ_V) are equivalent if $\phi_U = \phi_V$ on $U \cap V$.

Definition 2.2. A rational map $\phi : X \dashrightarrow Y$ is **dominant** if the image of U is dense in Y.

Definition 2.3. Two varieties X and Y are said to be **birational** if there is a rational map $\phi : X \dashrightarrow Y$ which admits an inverse $\psi : Y \dashrightarrow X$.

Example 2.4. The varieties $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 = Z(y_0y_1 - y_2y_3) \subset \mathbb{P}^3$ are birational.

Suppose $\mathbb{P}^2$ has coordinates $[x_0 : x_1 : x_2]$. We could consider the rational map defined on the locus $D(x_0) \cong \mathbb{A}^2_{x_1,x_2}$ given by $(x_1, x_2) \mapsto (x_1, x_2, x_1x_2) \subset \mathbb{A}^3_{y_0,y_1,y_2}$. The image is the affine variety $Z(y_0y_1 - y_2)$, which is $Z(y_0y_1 - y_2y_3) \cap D(y_3) \subset \mathbb{P}^3$.

On the affine pieces, this map is clearly invertible: $(x_1, x_2, x_1x_2) \mapsto (x_1, x_2)$. Therefore, it is a birational map and shows that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are birational.