ALGEBRAIC GEOMETRY: FRIDAY, FEBRUARY 17

1. Introduction to sheaves

Let Y be a variety, $p \in Y$, and U an open set containing p.

Definition 1.1. Define $\mathcal{O}(U) = \{\text{regular functions on } U\}$. This will be called the *structure sheaf* (once we know what sheaves are!).

Definition 1.2. The **local ring** of $p \in Y$ is

$$\mathcal{O}_{Y,p} = \{\text{germs of regular functions near } p\}.$$

This is defined with an equivalence relation: elements are pairs (U, f) where U is an open set and f is a regular function at p. If U, V are open sets containing p, $(U, f) = (V, g)$ if $f = g$ on $U \cap V$.

Observe: if $V \subset U \subset Y$, then there is restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$. We could alternately define the local ring at p as

$$\mathcal{O}_{Y,p} = \varinjlim_{p \in U} \mathcal{O}(U).$$

This is called the local ring as it is actually a local ring: the maximal ideal is the ideal of germs vanishing at p:

$$m_{Y,p} = \{f \in \mathcal{O}_{Y,p} \mid f(p) = 0\}$$

and given any $g \in \mathcal{O}_{Y,p}$ not in $m_{Y,p}$, then $g(p) \neq 0$ so $1/g$ is regular in some neighborhood of p, i.e. $g^{-1} \in \mathcal{O}_{Y,p}$. Consider the evaluation map $\mathcal{O}_{Y,p} \to k$ given by $f \mapsto f(p)$. The kernel is $m_{Y,p}$, so $\mathcal{O}_{Y,p}/m_{Y,p} \cong k$, i.e. $m_{Y,p}$ is maximal. (Can check: this is the unique maximal ideal.)

Definition 1.3. Given a variety Y, the **function field** of Y is

$$K(Y) = \{(U, f) \mid U \subset Y \text{ open }, f \in \mathcal{O}(U)\}/\sim$$

where $(U, f) \sim (V, g)$ if $f = g$ on $U \cap V$.
The elements of $K(Y)$ are **rational functions** on Y.

This is a field! Because Y is irreducible, any two open sets intersect, and addition and multiplication make sense (so it is a ring). For inverses, consider $(U, f) \in K(Y)$. Then, for any $f \neq 0$, let $V = U - U \cap Z(f)$, so $(V, 1/f) \in K(Y)$, and this is the inverse of (U, f).

Example 1.4. If $Y = \mathbb{A}^1$, $\mathcal{O}(Y) = k[x]$, and $\mathcal{O}_{Y,p} \cong k[x]_{(x)}$. (Why? Maybe this is intuitively clear, but we will prove it in general momentarily.)

Some other remarks:

Remark 1.5.
(1) There is always an embedding $\mathcal{O}_{Y,p} \hookrightarrow K(Y)$, and $K(Y) = \cup_{p \in Y} \mathcal{O}_{Y,p}$.
(2) The restriction map $\mathcal{O}(U) \to \mathcal{O}(V)$ is injective (if two functions agree on an open subset $V \subset U$, they agree on U by density of open sets). So, for all $p \in Y$, there is a natural injective restriction map $\mathcal{O}(Y) \hookrightarrow \mathcal{O}_{Y,p}$, and $\mathcal{O}(Y) = \cap_{p \in Y} \mathcal{O}_{Y,p}$.
(3) By definition of regular function and isomorphism, if Y and Y' are isomorphic varieties, then these rings $\mathcal{O}(Y)$, $\mathcal{O}_p$, $K(Y)$ are isomorphic to $\mathcal{O}(Y')$, $\mathcal{O}_{p'}$, $K(Y')$.

Before generalizing this construction, let us relate it to the rings we already know.

Theorem 1.6. *If $Y \subset \mathbb{A}^n$ is an affine variety, let $A(Y)$ be its affine coordinate ring. Then:*

(1) $\mathcal{O}(Y) \cong A(Y)$
(2) *For each $p \in Y$, define $m_p = \{f \in A(Y) \mid f(p) = 0\}$. The correspondence $p \mapsto m_p$ is a one-to-one correspondence between points of Y and maximal ideals in $A(Y)$.*
(3) *For each $p \in Y$, $\mathcal{O}_p \cong A(Y)_{m_p}$ and $\dim \mathcal{O}_p = \dim Y$.*
(4) *The field $K(Y)$ is isomorphic to the fraction field of $A(Y)$ (so, $K(Y)$ is a finitely generated field extension of k and $\mathrm{trdeg}_k K(Y) = \dim Y$.*

We'll use the following algebra theorem (from last week) in the proof; adding it here for convenience of the reader.

Theorem 1.7. *If k is a field and B is an integral domain that is finitely generated as a k-algebra, then:*

(1) $\dim B = \mathrm{trdeg}_k \mathrm{Frac}(B)$
(2) *For any prime ideal $I \subset B$, $\mathrm{height} I + \dim B/I = \dim B$.*

Now, onto the proof of the theorem.

Proof. We start with (2): let $\alpha : A(Y) \to \mathcal{O}(Y)$ be inclusion (each polynomial $f \in A(Y)$ is a regular function on Y), which is injective. From the correspondence between Z and I earlier, we know the points of Y (minimal algebraic subsets) correspond to maximal ideals of $A = k[x_1, \ldots, x_n]$ containing $I(Y)$, which are precisely the maximal ideals of $A(Y)$, and the correspondence exactly matches a point $p \in Y$ to $I(p) = \{f \in A(Y) \mid f(p) = 0\} = m_p$.

Now, to prove (3): for each $p \in Y$, there is a map (induced by α) $\alpha_p : A(Y)_{m_p} \to \mathcal{O}_p$, which is injective because α is injective, and surjective because, by definition, a regular function at p is f/g where $g(p) \neq 0$, i.e. $g \in A(Y) - m_p$, i.e. $f/g \in A(Y)_{m_p}$. Therefore, it is an isomorphism. This shows $A(Y)_{m_p} \cong \mathcal{O}_p$, and because m_p is maximal, $\dim \mathcal{O}_p = \mathrm{height} m_p$, so by the algebra theorem above (as $A(Y)/m_p = k$) we see that $\dim \mathcal{O}_p = \dim A(Y) = \dim Y$.

For (4): the fraction field of $A(Y)$ is isomorphic to the fraction field of $\mathcal{O}_p$ (for any p), which is just $K(Y)$ by definition. Because $A(Y)$ is finitely generated over k, his proves that $K(Y)$ is a finitely generated field extension of k, and again by the algebra fact, $\dim Y$ is the transcendence degree.

Finally, we prove (1). By the remark above, $\mathcal{O}(Y) \subset \cap_{p \in Y} \mathcal{O}_p$, and therefore

$$A(Y) \subset \mathcal{O}(Y) \subset \cap_{m_p} A(Y)_{m_p}.$$

But, because $A(Y)$ is an integral domain, it is equal to the intersection of its localizations at maximal ideas, so all three are equal and we have $A(Y) \cong \mathcal{O}(Y)$. □

This construction is an example of a *sheaf.*

Definition 1.8. Let X be a topological space. A **presheaf $\mathcal{F}$ of abelian groups on** X is the data of:

(1) For every open set $U \subset X$, an abelian group $\mathcal{F}(U)$, and $\mathcal{F}(\emptyset) = 0$
(2) For every $V \subset U$, there is a $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\rho_{UU} = id$ and if $W \subset V \subset U$, $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

Definition 1.9. Let X be a topological space. A **sheaf $\mathcal{F}$ on** X is a presheaf $\mathcal{F}$ satisfying the following additional conditions:

(1) For any open set $U \subset X$ and open cover $U = \cup V_\alpha$, if $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha}(s) = 0$ for each α, then $s = 0$. (If s 'restricts' to 0 on each open set, then is is 0.)

(2) For any open U and open cover $U = \cup V_\alpha$, if for each α there exists $s_\alpha \in \mathcal{F}(V_\alpha)$ such that $\rho_{V_\alpha(V\alpha \cap V_\beta)} s_\alpha = \rho_{V_\beta(V\alpha \cap V_\beta)} s_\beta$, then there exists $s \in \mathcal{F}(U)$ such that $\rho_{UV_\alpha})(s) = s_\alpha$. (If there exists a collection of s's on the open cover that agree on the overlaps, they can be 'glued' together to an s on the whole set.)

Example 1.10. Sheaves exist beyond the context of algebraic geometry. If X is a (smooth) manifold, we have several natural sheaves on X: the sheaf of continuous functions, the sheaf of differentiable functions, ..., the sheaf of infinitely differentiable functions, ...

Definition 1.11. If $\mathcal{F}$ is a sheaf on X and $U \subset X$ is an open set, an element $s \in \mathcal{F}(U)$ is called a **section** of $\mathcal{F}$ over U. An element $s \in \mathcal{F}(X)$ is called a **global section** of $\mathcal{F}$.

Definition 1.12. If X is a variety, the sheaf $\mathcal{O}_X$ is called the **structure sheaf** of X.

Example 1.13. On an affine variety X, the global sections of the structure sheaf $\mathcal{O}_X$ are $\mathcal{O}_X(X) = A(X)$ (polynomial functions on X).