ALGEBRAIC GEOMETRY: WEDNESDAY, FEBRUARY 15

1. Morphisms

Let X be a quasi-affine variety in $\mathbb{A}^n$.

Definition 1.1. A function $f : X \to k$ is **regular at** $p \in Y$ if there is an open neighborhood $p \in U$ and polynomials $g, h \in A = k[x_1, \ldots, x_n]$ such that $h \neq 0$ on U and $f = g/h$ on U. A function f **regular** on Y if it is regular at every point.

Example 1.2. Let $\overline{X} = Z(xy - zt) \subset \mathbb{A}^4$. Let $W = Z(y, t)$ and let $X = \overline{X} - W$.
Then, the function $f : X \to k$ defined by

$$f = \begin{cases} z/y & y \neq 0 \\ x/t & t \neq 0 \end{cases}$$

is a well-defined regular function on X.

Lemma 1.3. *A regular function is continuous (where k is thought of as $\mathbb{A}^1_k$ with the Zariski topology).*

Proof. Consider $f : X \to \mathbb{A}^1$. We must show f^{-1} of a closed set is closed. This is clear for $\emptyset$ and $\mathbb{A}^1$, and the only other closed sets of $\mathbb{A}^1$ are finite sets, so it suffices to show for a single point. Let $a \in \mathbb{A}^1$ and consider $f^{-1}(a) = \{p \in X \mid f(p) = a\}$.

Topology fact: if X is a topological space with open cover $X = \cup U_\alpha$ and $W \subset X$, then W is closed if and only if $W \cap U_\alpha$ is closed for each U_α.

By the topology fact, we consider a covering of X by open neighborhoods U_α such that on each U_α, $f = g_\alpha / h_\alpha$. But, on U_α,

$$\begin{aligned} f^{-1}(a) &= \{p \in X \mid f(p) = a\} \\ &= \{p \in X \mid g_\alpha(p)/h_\alpha(p) = a\} \\ &= \{p \in X \mid g_\alpha(p) - a h_\alpha(p) = 0\} \\ &= Z(g_\alpha - a h_\alpha) \end{aligned}$$

so $f^{-1}(a)$ is closed. □

We have a similar definition (and the same lemma) for quasi-projective varieties.

Definition 1.4. If X is quasi-projective, a function $f : X \to k$ is regular at $p \in X$ is there is an open neighborhood $p \in U$ and homogeneous polynomials $g, h \in S$ with $\deg g = \deg h$ such that $h \neq 0$ on U and $f = g/h$ on U. A function f is regular on X if it is regular at every point.

Remark 1.5. Suppose that f and g are regular functions on a variety (affine or projective) X, and $f = g$ on some open neighborhood U of X. Then, $Z(f - g)$ is closed but contains the dense set U, so $Z(f - g) = X$, and hence $f = g$ on X.

Definition 1.6. Suppose $k = \overline{k}$. The **category of varieties** over k is $\mathcal{V}ar$ whose objects are varieties (affine, quasi-affine, projective, or quasi-projective) and whose morphisms are continuous functions $\phi : X \to Y$ such that, for every open set $V \subset Y$ and for every regular function $f : V \to k$, the function $f \circ \phi : \phi^{-1}(V) \to k$ is regular.

Exercise 1.7. The composition of two morphisms is a morphism. This category is in fact a category.

Exercise 1.8. A function $\phi : X \to Y$ is a morphism if and only if for some open cover $Y = \cup V_\alpha$, for every regular function $f : V_\alpha \to k$, the function $f \circ \phi : \phi^{-1}(V_\alpha) \to k$ is regular.

Definition 1.9. A morphism $\phi : X \to Y$ is an **isomorphism** if there exists a morphism $\psi : Y \to X$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity functions.

Exercise 1.10. The functions $\phi_i : U_i = \mathbb{P}^n - Z(x_i) \to \mathbb{A}^n$ from last time are isomorphisms.

Example 1.11. Consider $Z(x^3 - y^2) \subset \mathbb{A}^2$. This is the image of the morphism $\phi : \mathbb{A}^1 \to \mathbb{A}^2$ given by $\phi(t) = (t^2, t^3)$. and ϕ is a homeomorphism but *not* an isomorphism: the inverse function is *not* a morphism (homework: check this!) and these two are not isomorphic.

Definition 1.12. A variety X is **affine** (resp. quasi-affine, projective, quasi-projective) if it is isomorphic to one.

Example 1.13. Let $V \subset \mathbb{A}^n$ be an affine variety and $f \in k[x_1, \ldots, x_n]$. Then, $X = V - Z(f)$ is affine. (But! X is not necessarily an affine variety in $\mathbb{A}^n$; it is just isomorphic to one.)

One example: Let $V = \mathbb{A}^1$ and $f = x$. Then, $X = V - Z(f)$ is the x-axis with the origin removed. This itself is not an affine variety, but it is isomorphic to the hyperbola $Z(xy - 1)$.

To prove X is affine in general: consider projection $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ given by

$$\pi(x_1, \ldots, x_{n+1}) \to (x_1, \ldots, x_n).$$

Let $Y \subset \mathbb{A}^{n+1}$ be $Y = Z(x_{n+1}f - 1) \cap \pi^{-1}(V)$ and let $\phi : X \to Y$ be

$$\phi(a_1, \ldots, a_n) = (a_1, \ldots, a_n, 1/f(a_1, \ldots, a_n)).$$

Note that π is the inverse of ϕ on X. Check that ϕ is a morphism to show that $X \cong Y$, which is affine.