ALGEBRAIC GEOMETRY: MONDAY, FEBRUARY 13

1. Projective Varieties

Let $k = \overline{k}$.

Definition 1.1. Projective n-space over k is

$$\mathbb{P}^n = (\mathbb{A}^{n+1} - (0, \ldots, 0))/\sim$$

where $\sim$ is the equivalence relation $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for $0 \neq \lambda \in k$.

A point $p \in \mathbb{P}^n$ is a choice of coordinates $(a_0, \ldots, a_n)$ in a given equivalence class, called *homogeneous* coordinates. We usually denote points in p by $p = [a_0 : \cdots : a_n]$.

This is the quotient of non-zero points in $\mathbb{A}^{n+1}$ where points on the same line through the origin are identified with each other.

To define varieties in this space, we will need to work with zero sets of polynomials that are invariant under this equivalence relation, which means we want only homogeneous polynomials: if $f(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree d, then $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$, so f being zero at $p = [a_0 : \cdots : a_n]$ is well-defined on the equivalence class of p.

To make the appropriate definitions, we first consider graded rings.

Definition 1.2. A **graded ring** S is a ring S with a decomposition

$$S = \oplus_{d \geq 0} S_d$$

where each S_d is an abelian group and for any d_1, d_2, $S_{d_1} \cdot S_{d_2} \subset S_{d_1+d_2}$.

An element $f \in S_d$ is called a **homogeneous element of degree** d. An ideal $I \subset S$ is a **homogeneous ideal** if $I = \oplus_{d \geq 0}(I \cap S_d)$.

Some commutative algebra facts:

Remark 1.3. An ideal I is homogeneous if and only if it is generated by homogeneous elements. Sums, products, intersections, and radicals of homogeneous ideals are homogeneous. Primality of homogeneous ideals is determined by considering homogeneous elements.

Let $S = k[x_0, \ldots, x_n]$. Then, S is a graded ring where S_d is the set of all homogeneous degree d polynomials. As we pointed out above, being in the zero locus of $f \in S_d$ is well-defined on equivalence classes $p = [a_0 : \cdots : a_n]$, so we can talk about the zero locus of polynomials (or sets of polynomials) in $\mathbb{P}^n$.

Definition 1.4. If f is a homogeneous polynomial $f \in S_d$, the **zero locus** of f is

$$Z(f) = \{p \in \mathbb{P}^n \mid f(p) = 0\}.$$

If T is any set of homogeneous elements of S, then the **zero locus** of T is

$$Z(T) = \{p \in \mathbb{P}^n \mid f(p) = 0 \forall f \in T\}.$$

If $I \subset S$ is a homogeneous ideal, let T be the set of all homogeneous elements in I (which is generated by a finite set since S is noetherian). We define $Z(I) = Z(T)$.

Definition 1.5. A subset $Y \subset \mathbb{P}^n$ is an algebraic set if $Y = Z(T)$ for some set T of homogeneous elements.

We leave it as an exercise to verify (just as we did for the affine case) that the Zariski topology defined by taking the closed subsets to be the algebraic sets is indeed a topology.

Definition 1.6. A **projective variety** is an irreducible algebraic set in $\mathbb{P}^n$. An open subset of a projective variety is a **quasi-projective variety.**

The **dimension** of a projective variety is its dimension as a topological space.

The **homogeneous ideal** of a projective variety Y is the ideal generated by

$$I(Y) = \langle \{f \in S \mid f \text{ homogeneous}, \ f(p) = 0 \forall p \in Y\} \rangle.$$

The **homogeneous coordinate ring** of Y is $S(Y) = S/I(Y)$.

Exercise 1.7. Prove that $\mathbb{P}^n$ is a noetherian topological space.

Exercise 1.8. Prove the Nullstellensatz for homogeneous ideals (which gives the same correspondence for Z and I as in the affine case, provided we throw out the irrelevant ideal, see Hartshorne Ch.1 Exercise 2.4).

It turns out that projective varieties are covered by affine varieties in the following way:

If $f \in S_1$ is a linear homogeneous polynomial, consider the zero set $Z(f)$ (called a hyperplane). To fix notation, if $f = x_i$ for $0 \leq i \leq n$, let $H_i = Z(x_i)$. Let $U_i = H_i^c = \mathbb{P}^n - U_i$. These sets U_i are commonly denoted $D(x_i)$ (the D here indicates 'doesn't vanish'). If $p \in \mathbb{P}^n$, $p = [a_0 : \cdots : a_n]$ with at least one $a_i \neq 0$, so $p \in U_i$ for some i, so the sets $U_i = D(x_i)$ form an open covering of $\mathbb{P}^n$.

Claim. For each i, $D(x_i)$ is homeomorphic to $\mathbb{A}^n$.

Consider the map

$$\phi_i : D(x_i) \to \mathbb{A}^n$$

defined by

$$\phi_i([a_0 : \cdots : a_n]) = (a_0/a_i, \ldots, \widehat{a_i/a_i}, \ldots, a_n/a_i).$$

This is well-defined because the ratio a_j/a_i is independent of choice of representative of the equivalence class of p.

Similarly, we have

$$\phi_i^{-1} : \mathbb{A}^n \to D(x_i)$$

given by

$$\phi_i^{-1}(b_1, \ldots, b_n) = [b_0 : \cdots : 1 : \cdots : b_n]$$

where the 1 is in the ith spot.

This shows ϕ_i is a bijection: if $a_i \neq 0$, we can choose a representative of p where $a_i = 1$, and in this case ϕ_i is

$$\phi_i([b_0 : \cdots : 1 : \cdots : b_n]) = (b_0, \ldots, \widehat{1}, \ldots, b_n).$$

To show it is a homeomorphism, we will show ϕ_i and ϕ_i^{-1} are closed maps. For notational simplicity, assume that $i = 0$.

Let S^h be the set of homogeneous elements of S. Define the function $\alpha : S^h \to A = k[y_1, \ldots, y_n]$ by $\alpha(f) = f(1, y_1, \ldots, y_n)$ and the function $\beta : A \to S^h$ by $\beta(g) = x_0^{\deg g} g(x_1/x_0, \ldots, x_n/x_0)$. To see that ϕ_0 is a homeomorphism, take any closed set $Y \subset U_0$ and its closure $\overline{Y} \subset \mathbb{P}^n$, which is an algebraic set $\overline{Y} = Z(T)$ for some $T \subset S^h$. Then, $\phi_0(Y) = Z(\alpha(T))$ which is a closed subset of $\mathbb{A}^n$. Similarly, if W is a closed subset of $\mathbb{A}^n$, $W = Z(T')$ and $\phi_0^{-1}(W) = Z(\beta(T')) \cap U$, which is a closed subset of U.

Therefore, $\mathbb{P}^n$ has an open cover by affine open sets isomorphic to $\mathbb{A}^n$. Furthermore, if Y is any projective variety, it has an open cover by affine varieties $Y \cap U_i$.

Example 1.9. We commonly use these functions α and β to convert projective varieties to affine varieties and vice versa.

For example, if $Y = V(xy - z^2) \subset \mathbb{P}^2$, then $\mathbb{P}^2$ is covered by three charts $D(x) \cup D(y) \cup D(z)$, and to get the corresponding variety on each chart, we set that variable 'equal to 1' via the map α (technically, we are changing the other two variables to the quotient by the first variable, but we usually don't write this). So, on $D(x)$, we get $Y \cap D(x) = V(y - z^2)$ which is a parabola, and on $D(z)$, we get $Y \cap D(z) = V(xy - 1)$ which is a hyperbola. To go from an affine variety to a projective one, we would homogeneize (think of the map β above): so, if we started with $V(xy - 1) \subset \mathbb{A}^2$, the corresponding object in $\mathbb{P}^2$ would be replacing x, y with $x/z, y/z$ and multiplying by z^2 to get $V(xy - z^2)$.