ALGEBRAIC GEOMETRY: FRIDAY, FEBRUARY 10

1. Dimension

Reminders from last time:

Theorem 1.1. *There is an inclusion-reversing bijection between algebraic sets $\{Y\}$ in $\mathbb{A}^n$ and radical ideals $\{J\}$ in A, given by $Y \mapsto I(Y)$ and $J \mapsto Z(J)$.*

Proposition 1.2. *An algebraic set Y is irreducible if and only if $I(Y)$ is a prime ideal.*

Example 1.3. Suppose $f \in A$ is an irreducible polynomial. Then (f) is prime, so $Y = Z(f) \subset \mathbb{A}^n$ is irreducible. If $n = 2$, then Y is called an affine *curve*. If $n = 3$, then Y is a *surface*. If $n > 3$, then Y is a *hypersurface*.

Example 1.4. If $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is a point, it is irreducible, so $I(p) = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal in A. (Minimal irreducible closed subset = maximal ideal.)

Definition 1.5. If $Y \subset \mathbb{A}^n$ is an algebraic set, then the **affine coordinate ring** of Y is the ring $A(Y) = A/I(Y)$.

The affine coordinate ring is the ring of functions on Y. Because $I(Y)$ is the ideal of functions vanishing on Y, any two elements of A differing by something in $I(Y)$ determine the same function on Y.

Note that, if Y is a variety (meaning it is an irreducible algebraic set), then $A(Y)$ is an integral domain because $I(Y)$ is prime, and it is a finitely generated k-algebra. Conversely, any finitely generated k-algebra B that is an integral domain can arise as the coordinate ring of some affine variety: write $B = k[x_1, \ldots, x_n]/J$ and let $Y = Z(J)$.

Definition 1.6. A topological space is **noetherian** if every descending chain of closed subsets stabilizes: for any sequence $Y_1 \supset Y_2 \supset \ldots$ of closed subsets, eventually we have $Y_r = Y_{r+1} = \ldots$.

Example 1.7. $\mathbb{A}^n$ is noetherian: given any $Y_1 \supset Y_2 \supset \ldots$, we have $I(Y_1) \subset I(Y_2) \subset \ldots$, but these are ideals in the noetherian ring A, so this stabilizes with $I(Y_r) = I(Y_{r+1}) = \ldots$, so by the correspondence between Y_i and $Z(I(Y_i))$, we have $Y_r = Y_{r+1} = \ldots$.

Proposition 1.8. *If X is noetherian, then every nonempty closed subset $Y \subset X$ can be written as $Y = Y_1 \cup \cdots \cup Y_r$ where each Y_i is an irreducible closed subset of X. Furthermore, if $Y_i \not\supset Y_j$ for all $i \neq j$, then this is unique up to reordering.*

Proof. Let $\mathcal{M} = \{\emptyset \neq Y \subset X \mid \nexists \text{ decomposition } Y = Y_1 \cup \cdots \cup Y_r : Y_i \text{ irreducible }\}$. Because X is noetherian, if $\mathcal{M}$ is not empty, it has a minimal element Y. Because $Y \in \mathcal{M}$, it is not irreducible, so $Y = Y_1 \cup Y_2$ and $Y_1, Y_2 \neq Y$. But, Y was the minimal element of $\mathcal{M}$, so $Y_1, Y_2 \notin \mathcal{M}$. Therefore, Y can be written as the union of the decomposition of Y_1 and Y_2, contradicting that $\mathcal{M}$ is not empty.

Now, suppose $Y = Y_1 \cup \cdots \cup Y_r = Z_1 \cup \cdots \cup Z_q$. Because $Y_i \subset Z_1 \cup \cdots \cup Z_q$ and Y_i is irreducible, we must have $Y_i \subset Z_j$ for some j. Similarly, $Z_j \subset Y_k$ for some k. Since $Y_i \not\subset Y_k$ for $i \neq k$ by assumption, this implies $Y_i = Y_k = Z_j$. This holds for any i, so we conclude the decompositions are equal up to reordering. □

Corollary 1.9. If $Y \subset \mathbb{A}^n$ is an algebraic set, then $Y = Y_1 \cup \cdots \cup Y_r$ for unique varieties Y_i. These are called the *irreducible components* of Y.

Definition 1.10. If X is a topological space, the *dimension* of X is

$$\dim X = \sup_m \{\exists \emptyset \neq Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m \subset X \mid Z_i \text{ irreducible}\}.$$

If X is an affine or quasi-affine variety, its dimension is defined to be its dimension as a topological space.

Example 1.11. $\dim \mathbb{A}^1 = 1$ because the only non-empty irreducible closed subsets are single points or the whole space, so the maximal chain we can have is $p \subset \mathbb{A}^1$ (for some $p \in \mathbb{A}^1$).

Exercise 1.12. If Y is an affine algebraic set, then $\dim Y$ (as a topological space) is equal to $\dim A(Y)$ (Krull dimension as a ring).

In other words, we can relation dimension of rings to dimension of varieties and algebraic sets, often via the following (the proof should be in Matsumura):

Theorem 1.13. *If k is a field and B is an integral domain that is finitely generated as a k-algebra, then:*

(1) $\dim B = \operatorname{trdeg}_k \operatorname{Frac}(B)$
(2) For any prime ideal $I \subset B$, $\operatorname{height} I + \dim B/I = \dim B$.

Corollary 1.14. $\dim \mathbb{A}^n = n$.

Proof. By the exercise, $\dim \mathbb{A}^n = \dim k[x_1, \dots, x_n]$, and by part (1) of the theorem, this is equal to n. ☐

Proposition 1.15. *If $Y \subset \mathbb{A}^n$ is quasi-affine, then* $\dim Y = \dim \overline{Y}$.

Proof. Consider a chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r \subset Y$. Then, taking the closure, we get

$$\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_r} \subset \overline{Y}$$

where $\overline{Z_i}$ is irreducible because Z_i is, and because $Z_i = \overline{Z_i} \cap Y$, $\overline{Z_i} \neq \overline{Z_j}$. So, $\dim Y \leq \dim \overline{Y}$.

This implies $\dim Y = n$ is finite, so choose a maximal length chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n \subset Y$. Because this is maximal, Z_0 must be a point p. Because each $Z_i \subset \overline{Z_i}$ is dense, taking the closure gives a maximal chain, and $p = \overline{Z_0}$ is a maximal ideal m in $A(\overline{Y})$. The $\overline{Z}_i$ for $i > 0$ correspond to prime ideals contained in m, so $\operatorname{height} m = n$, and $A(\overline{Y})/m = k$, so by the second part of the theorem above, $\dim \overline{Y} = n$. ☐

For certain affine algebraic sets, we can use this to further connect the algebra and the geometry.

Reminder of some results in commutative algebra:

Theorem 1.16 (Krull's Principal Ideal Theorem)**.** *If B is a noetherian ring and $f \in B$ is neither a zero divisor nor a unit, then every minimal prime ideal I containing f has height 1.*

Theorem 1.17. *A noetherian integral domain B is a UFD if and only if every prime ideal of height 1 is principal.*

These have a geometric meaning!

Theorem 1.18. *A variety Y in $\mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single non-constant irreducible polynomial $f \in A$.*

Proof. If f is irreducible, then $Z(f)$ is a variety and its ideal is $I = (f)$. By Krull's Principal Ideal Theorem, I has height 1, so by an earlier theorem, $Y = Z(f)$ has dimension $n-1$.

Now, suppose Y has dimension $n-1$. This corresponds to a prime ideal I of height 1, and A is a UFD, so $I = (f)$ is principal by the previous theorem. So, $Y = Z(f)$ and f is an irreducible polynomial. ☐

Exercise 1.19. $Y_1 = Z(y - x^2) \subset \mathbb{A}^2$ and $Y_2 = Z(xy - 1) \subset \mathbb{A}^2$ are 1-dimensional varieties. Show that $A(Y_1)$ is isomorphic to a polynomial ring in one variable over k but $A(Y_2)$ is not.

Exercise 1.20. Find a variety $Y \subset \mathbb{A}^3$ such that $\dim Y = 1$ but $I(Y)$ cannot be generated by 2 elements (so the previous theorem does not hold for varieties with higher codimension).