ALGEBRAIC GEOMETRY: MONDAY, FEBRUARY 6

Syllabus

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Office Hours: TBA; maybe MW after class (or by appointment).

Textbook(s): Our class will be based off of the following books:

(1) *Algebraic Geometry*, Hartshorne
(2) *The Rising Sea: Foundations of Algebraic Geometry*, Vakil
(3) *Basic Algebraic Geometry 1 and 2,* Shafarevich

I will assign homework every week or two weeks, depending on how quickly we get through the material. Your grade will be based on participation in class and your homework scores.

To learn algebraic geometry (or mathematics in general), you must *do* algebraic geometry. Doing the homework problems is an essential part of learning the material.

1. Introduction to Affine Varieties

Let $k = \overline{k}$ be an algebraically closed field.

Definition 1.1. **Affine n-space** is the set $\mathbb{A}^n_k = \{(a_1, a_2, \ldots, a_n) \mid a_i \in k\}$.

Let $A = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring in n variables over k. We consider A as the set of *functions* on $\mathbb{A}^n$: for any $f \in A$, $f : \mathbb{A}^n \to k$ is the function given by $f(p)$ (evaluating f at the point p).

Given any $f \in A$, we define

$$Z(f) = \{p \in \mathbb{A}^n \mid f(p) = 0\} \quad \text{the zeros of } f$$

and given any subset $T \subset A$, we define

$$Z(T) = \{p \in \mathbb{A}^n \mid f(p) = 0 \forall f \in T\}.$$

If $I = \langle T \rangle$ is the ideal generated by T, then we have $Z(T) = Z(I)$. Furthermore, because A is a noetherian ring, every ideal is finitely generated: $I = (f_1, f_2, \ldots, f_k)$, so $Z(T) = Z(f_1, f_2, \ldots, f_k)$.

Definition 1.2. A subset $Y \subset \mathbb{A}^n$ is an **algebraic set** if there exists some $T \subset A$ such that $Y = Z(T)$.

Proposition 1.3. *(1) $\emptyset$, $\mathbb{A}^n$ are algebraic sets.*
(2) If Y_1 and Y_2 are algebraic sets, then $Y_1 \cup Y_2$ is an algebraic set.
(3) If Y_α is a family of algebraic sets, then $\cap Y_\alpha$ is an algebraic set.

Corollary 1.4. The algebraic sets define a topology on $\mathbb{A}^n$ (the algebraic sets are the closed sets).

Proof. (1) Because $Z(1) = \emptyset$ and $Z(0) = \mathbb{A}^n$, $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.

(2) If $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$, let $T_1T_2 = \{fg \mid f \in T_1, g \in T_2\}$. Then, we can show that $Y_1 \cup Y_2 = Z(T1T2)$.

If $p \in Y_1 \cup Y_2$, suppose without loss of generality $p \in Y_1$. Then, for any $f \in T_1$, $f(p) = 0$, so for all $fg \in T_1T_2$, $fg(p) = f(p)g(p) = 0$. Therefore, $p \in Z(T_1T_2)$.

Now suppose $p \in Z(T_1T_2)$. If $p \in Y_1$, clearly $p \in Y_1 \cap Y_2$. So, suppose $p \notin Y_1$. Then, there exists $f \in T_1$ such that $f(p) \neq 0$. Because $p \in Z(T_1T_2)$, this means $g(p) = 0$ for all $g \in T_2$, so $p \in Z(T_2) = Y_2$. Therefore, $p \in Y_1 \cup Y_2$.

(3) For Y_α a family of algebraic sets, write $Y_\alpha = Z(T_\alpha)$. Then, $\cap Y_\alpha = Z(\cup T_\alpha)$.

□

Definition 1.5. The **Zariski topology** on $\mathbb{A}^n$ is the topology whose closed sets are the algebraic sets.

Example 1.6. The closed sets in $\mathbb{A}^1$ are $\emptyset$, $\mathbb{A}^n$, and any finite set.

The open sets are then $\emptyset$, $\mathbb{A}^n$, and complements of finite sets. In particular, this topology is not Hausdorff.

Definition 1.7. If X is a non-empty topological space, then X is **irreducible** if for all decompositions $X = X_1 \cup X_2$, where $X_1, X_2 \subset X$ closed, then either $X = X_1$ or $X = X_2$.

More casually: X can't be written as the union of two proper closed subsets.

Example 1.8. Because the only closed sets in $\mathbb{A}^1$ are finite and $\mathbb{A}^1$ is infinite (because k is algebraically closed), $\mathbb{A}^1$ is irreducible.

Lemma 1.9. *Any non-empty open set U of an irreducible space X is irreducible and dense.*

If Y is irreducible and $Y \subset X$, then $\overline{Y}$ is irreducible.

Proof. For the first statement, write $X = (X \setminus U) \cup \overline{U}$. Because $X \neq X \setminus U$ and X is irreducible, we must have $X = \overline{U}$. Therefore, U is dense. Now suppose $U = U_1 \cup U_2$, so $X = \overline{U} = \overline{U_1} \cup \overline{U_2}$. Because X is irreducible, we must have $X = \overline{U_i}$ for some $i \in \{1, 2\}$, so $U_i = U \cap \overline{U_i} = U$.

For the second statement, write $\overline{Y} = Z_1 \cup Z_2$. Then, $Y = (Z_1 \cap Y) \cup (Z_2 \cap Y)$, so for some $i \in \{1, 2\}$, $Y = Z_i \cap Y$, so $Y \subset Z_i$. Because Z_i is closed, this implies $\overline{Y} \subset Z_i$. □

Definition 1.10. An **affine algebraic variety** is an irreducible algebraic set in $\mathbb{A}^n$ (for some n).

A **quasi-affine algebraic variety** is an open subset of an affine algebraic variety.