Math 611 Midterm Practice

Midterm Exam will be on Thursday, 10/26, in class

Midterm policies:

- The midterm will cover chapters 1 - 5 of Dummit and Foote. If you are curious on whether or not you need to know a specific section/vocabulary word/formula, please ask!
- The midterm will consist of 5-7 problems that vary in difficultly. The problems will be similar to homework problems, problems assigned in lecture, or old qualifying exam problems. I have provided several practice problems below which will be similar to those on the actual exam. The first section of problems are less involved than the midterm problems will be, but you may want to start with them as a 'concept check'.
- You may bring a single-sided 8.5 $\times$ 11 inch handwritten notesheet (although this is not required). There are no restrictions on what can be written on the sheet.
- Exams will be graded and returned through Gradescope.

Small proofs/quick practice (the midterm problems will be more involved than these ones, but these are some practice problems that you can try as quick concept checks).

1. For x, y elements of a group G, prove that $\text{ord}(x) = \text{ord}(yxy^{-1})$. Use this to show that $\text{ord}(ab) = \text{ord}(ba)$ for all $a, b \in G$.
2. Find all numbers n such that S_7 contains an element of order n. What is the maximal order of an element of S_7? Find the smallest value of m such that $\sigma^m = 1$ for all $\sigma \in S_7$. (All parts of your answer should be proved/justified.)
3. Prove that $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$ are not isomorphic.
4. Let G be any group. Prove that the function $f : G \to G$ given by $f(g) = g^{-1}$ is a homomorphism if and only if G is abelian.
5. Let $\phi : G \to H$ be a homomorphism between groups G and H. Prove that $\ker \phi$ is a normal subgroup of G.
6. Prove that $\mathbb{Q}$ is not finitely generated.
7. Prove that, if $N \trianglelefteq G$ is a normal subgroup and $H \leq G$ is any subgroup, then $N \cap H \trianglelefteq H$.
8. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime dividing its order.
9. Show that the center of a direct product of groups is the direct product of the centers of the factors.
10. List all abelian groups of order 60, 75, and 100.

Midterm style problems.

1. Let G be a finite group and let $x, y \in G$ be distinct elements of order 2 that generate G. Prove that $G \cong D_{2n}$, where $n = |xy|$.
2. Let $x, y \in G$ be elements such that $\text{ord}(x) = n$ and $\text{ord}(y) = m$. If x and y commute, prove that $\text{ord}(xy)$ is equal to the least common multiple of n and m.

3. Let G be an abelian group and $D = \{(a,a) \mid a \in G\} \leq G \times G$ be the diagonal subgroup. Prove that D is normal in $G \times G$ and that $G \times G/D \cong G$.

4. Let $n \geq 3$. Prove that A_n contains a subgroup isomorphic to S_{n-2}.

5. Let $H \trianglelefteq G$ be a normal subgroup and let $g \in G$. Let C be the conjugacy class of $g \in G$. If $g \in H$, prove that $C \subset H$.

6. If $Z(G)$ has index n, prove that every conjugacy class in G has at most n elements.

7. Prove that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ by showing, for any $\sigma \in \mathrm{Aut}(G)$ and any $\phi_g \in \mathrm{Inn}(G)$ conjugation by g, then $\sigma\phi_g\sigma^{-1} = \phi_{\sigma(g)}$.

8. Suppose $|G| = 231$. Prove that G contains a normal Sylow 7- and 11-subgroup and that the Sylow 11-subgroup is contained in the center of $Z(G)$. (Possible hint for the last part: use the action of G on the subgroup by conjugation.)

9. Classify groups of order 28. (There are four isomorphism types.)

Sample midterm problems coming from qualifying exams (your midterm will include similar problems).

1. Show that there are exactly two isomorphism classes of non-abelian groups of order 8. Describe these two groups in terms of generators and relations. Show your work!

2. Let G be a group of order 60. Assume that the center $Z(G)$ has order divisible by 4. Show that G is abelian.

3. Show that there are no simple groups of order 80.

4. Show that there are at least two non-isomorphic, non-abelian groups of order 147.

5. Let p be a prime. Determine the number of conjugacy classes of a non-abelian group G of order p^3.

6. Let p be a prime and let H be the subset of upper triangular matrices in $GL_3(\mathbb{F}_p)$ with 1's on the diagonal.

   (a) Show that H is a p-Sylow subgroup of $GL_3(\mathbb{F}_p)$

   (b) Prove that the center of H is

   $$Z(H) = \left\{ \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid c \in \mathbb{F}_p \right\}.$$

   (c) Show that $H/Z(H) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.