DECEMBER 5 NOTES

1. 10.4: Tensor Products

Construction. Suppose that R is a subring of S, and that N is an R-module. Then, $S \times N$ is an abelian group. If N were to be an S module, we would have to define an action $S \times N \to N$ where $(s, n) \mapsto sn$ satisfying that $(s_1 + s_2)n = s_1 n + s_2 n$ and the rest of the module axioms. This doesn't quite work, but gives the inspiration for the construction.

Consider the free $\mathbb{Z}$-module $F(S \times N)$ on the set $S \times N$, which is the collection of all finite sums of elements (s_i, n_i) with $s_i \in S$ and $n_i \in N$. Let H be the subgroup generated by all elements of the form:

$$(s_1 + s_2, n) - (s_1, n) - (s_2, n)$$

$$(s, n_1 + n_2) - (s, n_1) - (s, n_2)$$

$$(sr, n) - (s, rn)$$

for elements $s, s_1, s_2 \in S$, $n, n_1, n_2 \in N$, and $r \in R$.

Denote by $S \otimes_R N$ ('S tensor N', where the symbol $\otimes$ is typeset by 'otimes') the quotient of $F(S \times N)$ by this subgroup H. Let $s \otimes n$ be the coset of the element (s, n) in this quotient. The group $S \otimes_R N$ is called the *tensor product* of S and N, elements of $S \otimes_R N$ are called *tensors* and elements of the form $s \otimes n$ are called *simple tensors*.

By construction, every element of the tensor product can be written as a finite sum of simple tensors, and we have forced the relations:

$$(s_1 + s_2) \otimes n = s_1 \otimes n + s_2 \otimes n$$

$$s \otimes (n_1 + n_2) = s \otimes n_1 + s \otimes n_2$$

$$sr \otimes n = s \otimes rn.$$

We define an action of S on $S \otimes_R N$ by

$$s(s_1 \otimes n_1 + \cdots + s_k \otimes n_k) = (ss_1) \otimes n_1 + \cdots + (ss_k) \otimes n_k.$$

One has to check that this is well defined (because there is typically no unique way of writing a tensor as a sum of simple tensors), but that follows by construction.

Finally, one can show that this action makes $S \otimes_R N$ into an S module. For example:

$$\begin{aligned}(s + s') \otimes (s_i, n_i) &= ((s + s')s_i) \otimes n_i \\ &= (ss_i + s's_i) \otimes n_i \\ &= (ss_i) \otimes n_i + (s's_i) \otimes n_i \\ &= s(s_i \otimes n_i) + s'(s_i \otimes n_i).\end{aligned}$$

The remaining axioms are checked similarly.

So, we have 'extended' the R-module N to the S-module $S \otimes_R N$. This is usually referred to as *extension of scalars*.

Note that there is a natural R-module homomorphism $i : N \to S \otimes_R N$ given by $i(n) = 1 \otimes n$. Using this homomorphism, we can show that module $S \otimes_R N$ is, in a precise sense, the 'smallest' S module we can make that admits a homomorphism from N. This is why this is usually referred to as 'extension' to S.

Theorem 1.1. *Let R be a subring of S and let N be an R-module. Let $i : N \to S \otimes_R N$ be the R-module homomorphism $i(n) = 1 \otimes n$. Suppose that L is any S-module and that $\phi : N \to L$ is any homomorphism of R-modules. Then, there exists a unique homomorphism of S-modules $\Phi : S \otimes_R N \to L$ such that $\phi = \Phi \circ i$. We express this with a diagram:*

$$\begin{array}{ccc} N & \xrightarrow{\;i\;} & S \otimes_R N \\ & \searrow^{\phi} & \downarrow \Phi \\ & & L \end{array}$$

Conversely, if $\Phi : S \otimes_R N \to L$ is any S-module homomorphism, $\phi = \Phi \circ i : N \to L$ is an R-module homomorphism.

Proof. Suppose $\phi : N \to L$ is an R-module homomorphism. There is a $\mathbb{Z}$-module homomorphism from $\Phi_F : F(S \times N)$ to L sending each generator (s, n) to $s\phi(n)$ (exercise: check this is a homomorphism). Because ϕ is an R-module homomorphism, the elements of H (the subgroup of relations with which we mod out $F(S \times N)$) must map to 0 under this homomorphism; for example:

$$\begin{aligned} \Phi_F((s_1 + s_2, n) - (s_1, n) - (s_2, n)) &= \Phi_F(s_1 + s_2, n) - \Phi_F(s_1, n) - \Phi_F(s_2, n) \\ &= (s_1 + s_2)\phi(n) - s_1\phi(n) - s_2\phi(n) \\ &= 0\phi(n) \\ &= 0. \end{aligned}$$

Therefore, this homomorphism Φ_F factors through the quotient $F(S \times N)/H = S \otimes_R N$. Let $\Phi : S \otimes_R N \to L$ be this homomorphism, which by definition is given by $\Phi(s \otimes n) = s\phi(n)$. This is actually a homomorphism of S-modules:

$$\begin{aligned} \Phi(s'(s_1 \otimes n_1) + (s_2 \otimes n_2)) &= \Phi(s'(s_1 \otimes n_1)) + \Phi(s_2 \otimes n_2) \\ &= \Phi((s's_1) \otimes n_1) + \Phi(s_2 \otimes n_2) \\ &= (s's_1)\phi(n_1) + s_2\phi(n_2) \\ &= s'(s_1\phi(n_1)) + s_2\phi(n_2) \\ &= s'\Phi(s_1 \otimes n_1) + \Phi(s_2 \otimes n_2). \end{aligned}$$

This homomorphism is unique because $S \otimes_R N$ is generated as an S-module by the elements $1 \otimes n$, and $\Phi(1 \otimes n) = \phi(n)$, so Φ is uniquely determined by ϕ.

The converse statement is automatic. □

The previous theorem is called a 'universal property' and controls the relationship between R-modules and S-modules.

Corollary 1.2. If $i : N \to S \otimes_R N$ is the homomorphism in the previous theorem, then $N/\ker i$ is the largest quotient of N that can be embedded in any S-module. In particular, if i is not injective, then N cannot be embedded in any S-module. (Here: 'embedding' means 'mapped injectively to'.)

Proof. By the First Isomorphism Theorem, $N/\ker i$ is mapped injectively to $S \otimes_R N$. If $\phi : N \to L$ is any homomorphism mapping the quotient $N/\ker \phi$ injectively to L, then this factors through the map $i : N \to S \otimes_R N$. Because $\ker i$ must be mapped to 0 by ϕ, this implies that $\ker i \subset \ker \phi$. Therefore, $N/\ker i$ is the largest quotient that can map injectively to an S-module L. □

Example 1.3. If R is any ring and N is any R-module, then $R \otimes_R N \cong N$ because we could consider $\phi : N \to N$ the identity map. Then, the diagram in the previous theorem says $\phi = \Phi \circ i : N \to N$, so $i : N \to R \otimes_R N$ is a bijective homomorphism with inverse Φ, so is an isomorphism.

In particular, if $R = \mathbb{Z}$ and A is any abelian group, then $\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A$.

Example 1.4. Let $R = \mathbb{Z}$. We just showed that $\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A$. What about if $S = \mathbb{Q}$? What is $\mathbb{Q} \otimes_{\mathbb{Z}} A$? First observe that $s \otimes 0 = s \otimes (0 + 0) = s \otimes 0 + s \otimes 0$, so subtracting one $s \otimes 0$ from both sides shows that $s \otimes 0 = 0$. (This is true in any tensor product!)

Now suppose A is a finite abelian group with $|A| = n$. By Lagrange's Theorem, this means $na = 0$ for any $a \in A$. Let $q \otimes a \in \mathbb{Q} \otimes_{\mathbb{Z}} A$ be any simple tensor. Because $q = (q/n)n$, we can write

$$q \otimes a = ((q/n)n) \otimes a = q/n \otimes na = q/n \otimes 0 = 0$$

so any simple tensor is just equal to 0. Because every element of $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is a sum of simple tensors, this implies that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ for all finite abelian groups.

Now we define tensor products in general, where we just replace S with any R-module. Let M and N be two R-modules. Because R is commutative, let us define a *right* action on M as $mr = rm$. (As in, the right action is the same as the left action.) This makes M into an R-module with a right action: $m(rs) = (rs)m = (sr)m = s(rm) = (rm)s = m(rs)$.

Consider again the free $\mathbb{Z}$ module $F(M \times N)$ and H the subgroup generated by

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n)$$
$$(m, n_1 + n_2) - (m, n_1) - (m, n_2)$$
$$(mr, n) - (m, rn)$$

for elements $m, m_1, m_2 \in S$, $n, n_1, n_2 \in N$, and $r \in R$ and define $M \otimes_R N = F(M \times N)/H$. Again, the elements are called tensors, and the elements of the form $m \otimes n$ are called simple tensors. By defining the action of R to be $r(m \otimes n) = (rm) \otimes n = (mr) \otimes n = m \otimes (rn)$, we see that $M \otimes_R N$ is an R-module.

Definition 1.5. Suppose R is commutative and M, N are R-modules. Define the right action on M as above. For any abelian group L, a map $\phi : M \times N \to L$ is called **balanced** if

$$\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$$
$$\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$$
$$\phi(m, rn) = \phi(mr, n).$$

If L is an R-module, the map ϕ is called R-**bilinear** if it is balanced and $\phi(mr, n) = \phi(m, rn) = r\phi(m, n)$.

Exactly as before for $S \otimes_R N$, we have a universal property. The map $i : M \times N \to M \otimes_R N$ given by $i(m, n) = m \otimes n$ is R-bilinear, and given any R-bilinear map $\phi : M \times N \to L$, there is a unique map $\Phi : M \otimes_R N \to L$ such that $\phi = \Phi \circ i$:

$$\begin{array}{ccc} M \times N & \xrightarrow{\ i\ } & M \otimes_R N \\ & \searrow^{\phi} & \downarrow \Phi \\ & & L \end{array}$$

Time for examples!

Example 1.6. The tensor product $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3$ is 0 because $3a = a$ for any $a \in \mathbb{Z}_2$, so

$$a \otimes b = 3a \otimes b = a \otimes 3b = a \otimes 0 = 0.$$

In general, $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$ where $d = \gcd(n, m)$: let $a \otimes b$ be any element in the tensor product. Then, $a \otimes b = a \otimes (b1) = ab \otimes 1 = (ab)(1 \otimes 1)$, so any tensor is a multiple of $1 \otimes 1$, so $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$ is cyclic. Furthermore, since $d = xn + ym$ for some integers x, y, we have $d(1 \otimes 1) = (xn + ym)(1 \otimes 1) = xn(1 \otimes 1) + ym(1 \otimes 1) = x(n \otimes 1) + y(1 \otimes m) = 0 + 0 = 0$, so the order of this element $1 \otimes 1$ is a divisor of d. By considering the map $\mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{Z}_d$ mapping $(a, b) \to ab$

(mod d) which is $\mathbb{Z}$-bilinear, it factors through the tensor product and the induced map sends $1 \otimes 1$ to 1, which has order d, so $1 \otimes 1$ has order at least d. Therefore, $1 \otimes 1$ has order exactly d and $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_d$.

Example 1.7. Let R be a ring and I an ideal and N an R-module. There is an N-submodule IN defined to be all finite sums of products of elements in I with elements in N.

In this case, $R/I \otimes_R N \cong N/IN$. Because $1 \in R/I$ (the image of 1 in R) generates R/I as an R-module, the elements $1 \otimes n$ generate $R/I \otimes_R N$. There is an R-module homomorphism $N \to R/I \otimes_R N$ sending $n \mapsto 1 \otimes n$, which is surjective because the elements $1 \otimes n$ generate the tensor product. The kernel must contain IN because, if $a_i n_i \in IN$ $(a_i \in I, n_i \in N)$, this maps to $1 \otimes a_i n_i = a_i \otimes n_i = 0$ because $a_i = 0 \in R/I$. This gives a surjective map $N/IN \to R/I \otimes_R N$, and we must show it is an isomorphism. But this follows because it has an inverse: $R/I \otimes_R N \to N/IN$ given by $(r, n) \mapsto rn$ can be checked to be the inverse of this map.

Remark 1.8. The previous example is a special case of our first construction of $S \otimes_R N$, where $S = R/I$ (because there is a natural homomorphism $f : R \to R/I$). So, this tensor product $R/I \otimes_R N$ is an R-module, but it is *also* an R/I-module as constructed above. In general, if R, S are commutative rings and M is both an R and an S module (called an R, S-bimodule) and N is an R-module, then $M \otimes_R N$ will also be an R, S-bimodule.

A few other properties of the tensor product (we ran out of time for these in class, but take a look if you wish):

Theorem 1.9. *If $\phi : M \to M'$ and $\psi : N \to N'$ are R-module homomorphisms, then*

$$\phi \otimes \psi : M \otimes_R N \to M' \otimes_R N'$$

given by $(\phi \otimes \psi)(m \otimes n) = \phi(m) \otimes \phi(n)$ is an R-module homomorphism.

Theorem 1.10. *If M, N, L are R-modules, then $(M \otimes_R N) \otimes_R L \cong M \otimes_R (N \otimes_R L)$.*

Theorem 1.11. *Let M, M' and N, N' be R-modules. Then,*

$$(M \oplus M') \otimes_R N \cong (M \otimes_R N) \oplus (M' \otimes_R N)$$
$$M \otimes_R (N \oplus N') \cong (M \otimes_R N) \oplus (M \otimes_R N').$$

Corollary 1.12. If S is a ring and $f : R \to S$ a homomorphism so that S is an R-module via $rs = f(r)s$, then $S \otimes_R \cong R$ and $S \otimes_R R^n \cong S^n$.

Corollary 1.13. For a ring R, $R^s \otimes_R R^t \cong R^{st}$. This says that the tensor product of two free modules of finite rank is again free.